Instantaneous Gratification: Online Appendices

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This document contains the online appendices for Harris & Laibson "Instantaneous Gratification," Quarterly Journal of Economics (forthcoming):

Appendix E: The Alternative Approach to Deriving the IG Model

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E. The Alternative Approach to Deriving the IG Model

In the PF model, there is a sequence of selves $\{0, 1, 2, ...\}$, each of whom has a strictly positive span of control. In the IG model there is a continuum of selves $[0, \infty)$, each of whom has an infinitesimal span of control. In formulating the objective of the IG consumer, it is therefore important to bear in mind that her span of control is an instant, and that changes in her behavior have only an infinitesimal effect on her objective. Careful track must be kept of such infinitesimal effects.

Consider self $s \in [0, \infty)$, and suppose that all future selves use the consumption function $\widetilde{c} : [0, \infty) \to (0, \infty)$. Then the continuation-value function of self s is exactly the same as the continuation-value function of the PF consumer, namely v. In particular, v satisfies the differential equation

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - \widetilde{c}\,)\,v' - \gamma\,v + u(\,\widetilde{c}\,) \tag{36}$$

for $x \in [0, \infty)$, where we have suppressed the dependence of v and $\widetilde{c}$ on x.

Suppose further that self s has wealth x, and that she chooses the consumption level $c \in (0, \infty)$. Then the current value of self s is

$$w(x) = \mathrm{E}_s\big[u(c)\,dt + \beta\,\exp(-\gamma\,dt)\,v(x + dx)\big].$$

Now, Itô's Lemma implies that $\exp(-\gamma\,dt) = 1 - \gamma\,dt$ and

$$v(x + dx) = v(x) + v'(x)\,dx + \tfrac{1}{2}\,v''(x)\,(dx)^2.$$

Moreover $dx = (\mu\,x + y - c)\,dt + \sigma\,x\,dz$ and $(dx)^2 = \sigma^2\,x^2\,dt$. Hence

$$\begin{aligned} w(x) &= \mathrm{E}_s\left[u(c)\,dt + \beta\,v(x) + \beta\,\Big(v'(x)\,dx + \tfrac{1}{2}\,v''(x)\,(dx)^2 - \gamma\,v(x)\,dt\Big)\right] \\ &= \beta\,v(x) + \Big(\beta\,\big(\tfrac{1}{2}\,\sigma^2\,x^2 v''(x) + (\mu\,x + y - c)\,v'(x) - \gamma\,v(x)\big) + u(c)\Big)\,dt. \end{aligned}$$

In other words, there are two contributions to the current value of self s: the non-infinitesimal contribution $\beta\,v(x)$, and the infinitesimal contribution

$$\Big(\beta\,\big(\tfrac{1}{2}\,\sigma^2\,x^2 v''(x) + (\mu\,x + y - c)\,v'(x) - \gamma\,v(x)\big) + u(c)\Big)\,dt. \tag{37}$$

It follows at once that $w(x) = \beta\, v(x)$.

Furthermore the infinitesimal contribution (37) depends on c only via the term

$$(u(c) - \beta\, v'(x)\, c)\, dt.$$

Hence, in order to maximize her current value, self s need only choose c to maximize this expression. Bearing in mind that self s is free to choose any $c \in (0, \infty)$ when $x > 0$, and that she must choose $c \in (0, y]$ when $x = 0$, it follows that c must satisfy the optimality condition

$$\left\{ \begin{array}{ll} u'(c) = \beta\, v' & \text{if } x > 0 \\ u'(c) = \max\{u'(y), \beta\, v'\} & \text{if } x = 0 \end{array} \right\}, \tag{38}$$

where we have suppressed the dependence of v' on x.

Next, assuming that u is bounded below, then – just as in Section 3.3 – v is bounded below by $\frac{1}{\gamma}\, u(0)$. Furthermore v is bounded above by the value function $\overline{v}$ of a consumer who: (i) has utility function u; and (ii) discounts the future exponentially at rate γ. Hence

$$\tfrac{1}{\gamma}\, u(0) \leq v \leq \overline{v} \tag{39}$$

for all $x \in [0, \infty)$.

Next, in a stationary equilibrium we must have $c = \widetilde{c}$. Using this observation to eliminate $\widetilde{c}$ from (36), we arrive at the following definition:

Definition 12. *Suppose that u is bounded below. Then the* ***Bellman equation of the IG consumer with global lower bound $\frac{1}{\gamma}\, u(0)$*** *consists of the differential equation*

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, v'' + (\mu\, x + y - c)\, v' - \gamma\, v + u(c) \tag{40}$$

for all $x \in [0, \infty)$, the optimality condition

$$\left\{ \begin{array}{ll} u'(c) = \beta\, v' & \textit{if } x > 0 \\ u'(c) = \max\{u'(y), \beta\, v'\} & \textit{if } x = 0 \end{array} \right\} \tag{41}$$

and the global bounds

$$\tfrac{1}{\gamma}\, u(0) \leq v \leq \overline{v} \tag{42}$$

for all $x \in [0, \infty)$.

Notice that the difference between the earlier Definition 3 (of the Bellman equation of the IG consumer) and this new Definition 12 (of the Bellman equation of the IG consumer with global lower bound $\frac{1}{\gamma}\,u(0)$) is that the global lower bound in Definition 3 (namely $\frac{1}{\gamma}\,u(y)$) has been replaced by the weaker global lower bound $\frac{1}{\gamma}\,u(0)$. The final step is therefore to show that, if u is bounded below, then any solution of the Bellman equation of the IG consumer with global lower bound $\frac{1}{\gamma}\,u(0)$ is in fact a solution of the Bellman equation of the IG consumer.

This is easily done. Indeed, provided that u is bounded below, the whole of the existence and uniqueness machinery developed in Section 5 applies to the Bellman equation of the IG consumer with global lower bound $\frac{1}{\gamma}\,u(0)$. In particular, any solution of the Bellman equation of the IG consumer with global lower bound $\frac{1}{\gamma}\,u(0)$ is a solution of the Bellman equation of the $\widehat{u}$ consumer with global lower bound $\frac{1}{\gamma}\,u(0)$. Now, since the $\widehat{u}$ consumer is a straight optimizer, it is easy to see that the value function of the $\widehat{u}$ consumer must in fact satisfy the tighter global lower bound $\frac{1}{\gamma}\,u(y)$. Overall, then, any solution of the Bellman equation of the IG consumer with global lower bound $\frac{1}{\gamma}\,u(0)$ is a solution of the Bellman equation of the IG consumer, as required.

F. SOLUTION OF THE PF MODEL WITH $y = 0$

Substituting for v and c in equation (21) and equating the constant term to 0, we get $\Theta = \frac{1}{\gamma}$. Equation (21) then simplifies to

$$0 = \mu - \alpha - \tfrac{1}{2}\,\rho\,\sigma^2 + \gamma\,u\!\left(\frac{\alpha}{\theta}\right). \tag{43}$$

Second, substituting for v, w and c in equation (22) and equating the constant term to 0, we get $\Phi = \frac{\gamma+\beta\,\lambda}{\gamma\,(\gamma+\lambda)}$. Equation (22) then simplifies to

$$0 = \mu - \alpha - \tfrac{1}{2}\,\rho\,\sigma^2 + (\gamma+\lambda)\left(\frac{\gamma}{\gamma+\beta\,\lambda}\,u\!\left(\frac{\alpha}{\phi}\right) + \frac{\beta\,\lambda}{\gamma+\beta\,\lambda}\,u\!\left(\frac{\theta}{\phi}\right)\right). \tag{44}$$

Last, substituting for w and c in equation (23), we get

$$u'(\alpha) = \frac{\gamma+\beta\,\lambda}{\gamma\,(\gamma+\lambda)}\,\phi\,u'(\phi). \tag{45}$$

Now, for all ρ, u satisfies the functional equation $(1-\rho)\,u(z) = z\,u'(z) - 1$. Hence, multiplying equation (43) through by $1-\rho$, putting $m(z) = z\,u'(z)$ and rearranging, we obtain

$$\frac{m(\alpha)}{m(\theta)} = \frac{\gamma - (1-\rho)\,(\mu - \alpha - \frac{1}{2}\,\rho\,\sigma^2)}{\gamma}. \tag{46}$$

Similarly, from equation (44), we obtain

$$\frac{\gamma}{\gamma + \beta\,\lambda}\,\frac{m(\alpha)}{m(\phi)} + \frac{\beta\,\lambda}{\gamma + \beta\,\lambda}\,\frac{m(\theta)}{m(\phi)} = \frac{\gamma + \lambda - (1-\rho)\,(\mu - \alpha - \frac{1}{2}\,\rho\,\sigma^2)}{\gamma + \lambda}. \tag{47}$$

Last, multiplying (45) through by α and dividing through by $\phi\,u'(\phi)$, we obtain

$$\frac{m(\alpha)}{m(\phi)} = \frac{\gamma + \beta\,\lambda}{\gamma\,(\gamma + \lambda)}\,\alpha. \tag{48}$$

Using equations (46) and (48), we can eliminate $\frac{m(\alpha)}{m(\phi)}$ and $\frac{m(\theta)}{m(\phi)} = (\frac{m(\alpha)}{m(\phi)})/(\frac{m(\alpha)}{m(\theta)})$ from (47) to obtain the quadratic (24) given in the main text, namely

$$0 = \frac{\lambda}{1+\lambda}\,\left((\rho + \beta - 1)\,\alpha - \widetilde{\gamma}\right) + \frac{1}{1+\lambda}\left(\rho\,(1-\rho)\,\alpha^2 + (2\,\rho - 1)\,\widetilde{\gamma}\,\alpha - \widetilde{\gamma}^2\right),$$

where

$$\widetilde{\gamma} = \gamma - (1-\rho)\,(\mu - \tfrac{1}{2}\,\rho\,\sigma^2).$$

This quadratic is a convex combination of the affine term

$$(\rho + \beta - 1)\,\alpha - \widetilde{\gamma}$$

and the quadratic term

$$\rho\,(1-\rho)\,\alpha^2 + (2\,\rho - 1)\,\widetilde{\gamma}\,\alpha - \widetilde{\gamma}^2.$$

Moreover the quadratic term is convex when $\rho \le 1$ and concave when $\rho \ge 1$.

In the case in which $\rho < 1$, one can take advantage of the convexity of the quadratic term to show that there are two solutions of (24). The first is always positive, varying from $\frac{\widetilde{\gamma}}{\rho}$ when $\lambda = 0$ to $\frac{\widetilde{\gamma}}{\rho+\beta-1}$ when $\lambda = \infty$. The second is always negative, varying from $-\frac{\widetilde{\gamma}}{1-\rho}$ when $\lambda = 0$ to $-\infty$ when $\lambda = \infty$. Since the second solution gives rise to a negative average propensity to consume, the first solution is the only relevant one.

In the case in which $\rho = 1$, the quadratic term degenerates into an affine term and the unique solution of (24) is $\frac{\tilde{\gamma}\,(\tilde{\gamma}+\lambda)}{\tilde{\gamma}+\beta\,\lambda}$. This varies from $\widetilde{\gamma}$ when $\lambda = 0$ to $\frac{\widetilde{\gamma}}{\beta}$ when $\lambda = \infty$.

In the case in which $\rho > 1$, one can take advantage of the concavity of the quadratic term to show that there are again two solutions of (24). Both solutions are always positive. The first varies from $\frac{\tilde{\gamma}}{\rho}$ when $\lambda = 0$ to $\frac{\tilde{\gamma}}{\rho+\beta-1}$ when $\lambda = \infty$. The second varies from $\frac{\tilde{\gamma}}{\rho-1}$ when $\lambda = 0$ to $+\infty$ when $\lambda = \infty$. Since the right-hand side of equation (46) can be written in the form $\frac{\tilde{\gamma}-(\rho-1)\,\alpha}{\gamma}$, the second solution would force $m(\theta) \leq 0$ (with equality iff $\lambda = 0$). The first solution is therefore the only relevant one.

Finally, note that the relevant solution of the quadratic can be written in the form

$$\alpha = \frac{2\,\rho\,\tilde{\gamma} + \lambda\,(\beta+\rho-1) - \tilde{\gamma} - \sqrt{(\lambda\,(\beta+\rho-1)-\tilde{\gamma})^2 + 4\,\lambda\,\beta\,\rho\,\tilde{\gamma}}}{2\,\rho\,(\rho-1)}.$$

Moreover equations (46) and (48) yield

$$m(\theta) = \frac{\gamma\,m(\alpha)}{\tilde{\gamma} + (1-\rho)\,\alpha}, \quad m(\phi) = \frac{\gamma\,(\gamma+\lambda)\,m(\alpha)}{(\gamma+\beta\,\lambda)\,\alpha}.$$

The behavior of the value functions $v(x) = \frac{1}{\gamma}\,u(\theta\,x)$ and $w(x) = \frac{\gamma+\beta\,\lambda}{\gamma\,(\gamma+\lambda)}\,u(\phi\,x)$ as a function of λ can therefore be deduced from that of α.

G. Proof of Theorems 9, 10 and 11: Characterization of the Consumption Function in the Case $y > 0$

In this appendix, we outline the proof of Theorems 9, 10 and 11.

G.1. Some background information. In this section we begin by recalling that, by definition of equilibrium in the IG model, the value function v satisfies the global bounds

$$\tfrac{1}{\gamma}\,u(y) \leq v \leq \overline{v} \tag{49}$$

for all $x \in [0, \infty)$, where $\overline{v}$ is the value function of a consumer who: (i) has utility function u; and (ii) discounts the future exponentially at rate γ. The main significance of these bounds for our current purposes is that $\overline{v}$ is strictly concave, and therefore v cannot be convex.

We now state without proof two results that will help to organize the subsequent discussion on the form of the consumption function. Our first result concerns the smoothness of v.

Proposition 13. *Suppose that $\beta < 1$. Then v is infinitely differentiable on $[0, \infty)$.* ■

In particular, the discontinuity in $\widehat{u}$ at $x = 0$ (i.e. the fact that $\widehat{u}_+ \neq \widehat{u}_0$) does not translate into a discontinuity in v or any of its derivatives at $x = 0$. On the contrary, Proposition 13 actually depends on this discontinuity: when $\beta = 1$ (and therefore $\widehat{u}_+ = \widehat{u}_0$), v is not smooth at $x = 0$ when $\mu < \gamma$. The discontinuity in $\widehat{u}$ at $x = 0$ does, however, give rise to a different kind of discontinuity: as we shall see below, $v'(0)$ does not always vary continuously with μ. In fact, there exists $\mu_1 \in (\gamma, \overline{\mu})$ such that $v'(0)$ jumps up from $v'_L = \widehat{u}'_+(\psi\, \overline{c}) < \widehat{u}'_+(y)$ to $v'_R = \widehat{u}'_+(\psi\, y) > \widehat{u}'_+(y)$ as μ crosses μ_1. (Recall that $\psi = \frac{\rho-(1-\beta)}{\rho}$ and that $\overline{c}$ is the unique solution of the equation $u'(\overline{c}) = \beta\, \frac{u(\overline{c})-u(y)}{\overline{c}-y}$.)

Our second result states that the shadow value of wealth is always strictly positive:

Proposition 14. $v' > 0$ *on* $[0, \infty)$. ■

This is economically obvious: the $\widehat{u}$ consumer can always consume more in its current span of control.

G.2. A mathematical intuition. Recall that the utility function of the $\widehat{u}$ consumer has two parts: $\widehat{u}(\widehat{c}, x) = \widehat{u}_0(\widehat{c})$ when $x = 0$; and $\widehat{u}(\widehat{c}, x) = \widehat{u}_+(\widehat{c})$ when $x > 0$. Moreover $\widehat{u}_0(\widehat{c}) \geq \widehat{u}_+(\widehat{c})$ for all $\widehat{c} \in (0, y]$, with strict inequality when $\widehat{c} \in (\psi\, y, y]$. (See Figure 4.) In other words, the $\widehat{u}$ consumer obtains a utility premium when $x = 0$.

This suggests that, at any given wealth level, the $\widehat{u}$ consumer must choose between two strategies. The first, high-consumption, strategy is to dissave until her wealth runs out, and then enjoy the utility premium that she obtains at $x = 0$. The second, low-consumption, strategy is to save forever in order to take advantage of the higher asset income associated with higher financial wealth. Which of these two strategies is better will depend on μ. If μ is low, then the high-consumption strategy will be better no matter how large the wealth of the $\widehat{u}$ consumer. Similarly, if μ is high, then the low-consumption strategy will be better no matter how small the wealth of the $\widehat{u}$ consumer. However, if μ is intermediate then the high-consumption strategy will be better when wealth is low (and therefore the utility premium will be enjoyed after a relatively short wait) and the low-consumption strategy will be better when wealth is high (and therefore the prospect of the utility premium is too distant). Moreover consumption may in principle decrease with wealth over an intermediate range of wealth levels, as the $\widehat{u}$ consumer adjusts from the high-consumption strategy associated with low wealth to the low-consumption strategy associated with high wealth.

G.3. The boundary condition at $x = 0$. The value function v must satisfy two related conditions at $x = 0$. To derive the first of these conditions, note that the Bellman equation of the $\widehat{u}$ consumer takes the form

$$0 = \tfrac{1}{2}\,\sigma^2\, x^2\, v'' + (\mu\, x + y)\, v' - \gamma\, v + \widehat{h}_+(v') \tag{50}$$

for $x > 0$. Letting $x \downarrow 0$ in this equation, taking advantage of Proposition 13 and rearranging yields

$$v(0) = \tfrac{1}{\gamma}\left(y\, v'(0) \, + \widehat{h}_+(v'(0))\right). \tag{51}$$

The second of these conditions is simply the Bellman equation of the $\widehat{u}$ consumer at $x = 0$ which, on rearrangement, becomes

$$v(0) = \tfrac{1}{\gamma}\left(y\, v'(0) + \widehat{h}_0(v'(0))\right). \tag{52}$$

Figures 6a, 6b and 6c illustrate the locus of points $(v'(0), v(0))$ satisfying equation (51), the locus of points $(v'(0), v(0))$ satisfying equation (52) and the locus of points $(v'(0), v(0))$ satisfying both equations.

As Figure 6c shows, there are two possible boundary configurations. First, the $\widehat{u}$ consumer may opt for the utility premium and set $\widehat{c}(0) = y$. In this case $v(0) = \frac{1}{\gamma}\,\widehat{u}_0(y)$, and $v'(0)$ must take on the low value $v'_L = \widehat{u}'_+(\psi\,\overline{c}) < \widehat{u}'_+(y)$ in order to justify the $\widehat{u}$ consumer's high consumption level for small $x > 0$. Second, the $\widehat{u}$ consumer may forgo the utility premium and set $\widehat{c}(0) \leq \psi\, y$. In this case $v(0) \geq \frac{1}{\gamma}\,\widehat{u}_0(y)$, and we must have $v'(0) \geq v'_R = \widehat{u}'_+(\psi\, y) > \widehat{u}'_+(y)$ in order to justify the $\widehat{u}$ consumer's low consumption level at $x = 0$. We refer to these two configurations as the low-shadow-value and high-shadow-value boundary configurations respectively.

Our next major objective is to show that there exists $\mu_1 \in (\gamma, \overline{\mu})$ such that the low-shadow-value boundary configuration occurs when $\mu < \mu_1$ and the high-shadow-value boundary configuration occurs when $\mu > \mu_1$. To this end, we shall need several supporting results.

G.4. Once convex, always strictly convex. The following result only uses the fact that v satisfies the Bellman equation of the $\widehat{u}$ consumer in the interior of the wealth space.

Proposition 15. *Suppose that $\mu < \gamma$ and that either*

1. *there exists $x_0 \geq 0$ such that $v''(x_0) > 0$, or*

Figure 6a: graph of $v(0) = \frac{1}{\gamma}\left(y\, v'(0) + \hat{h}_+(v'(0))\right)$

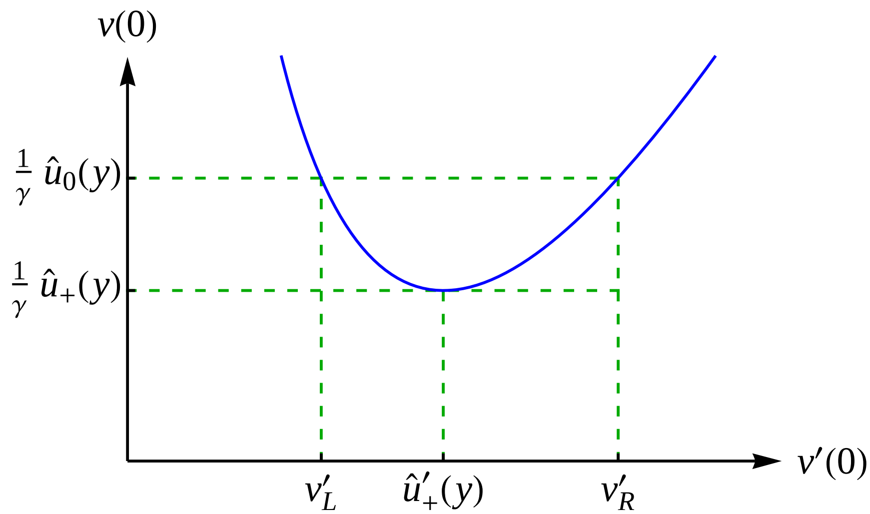


Figure 6b: graph of $v(0) = \frac{1}{\gamma}\left(y\, v'(0) + \hat{h}_0(v'(0))\right)$

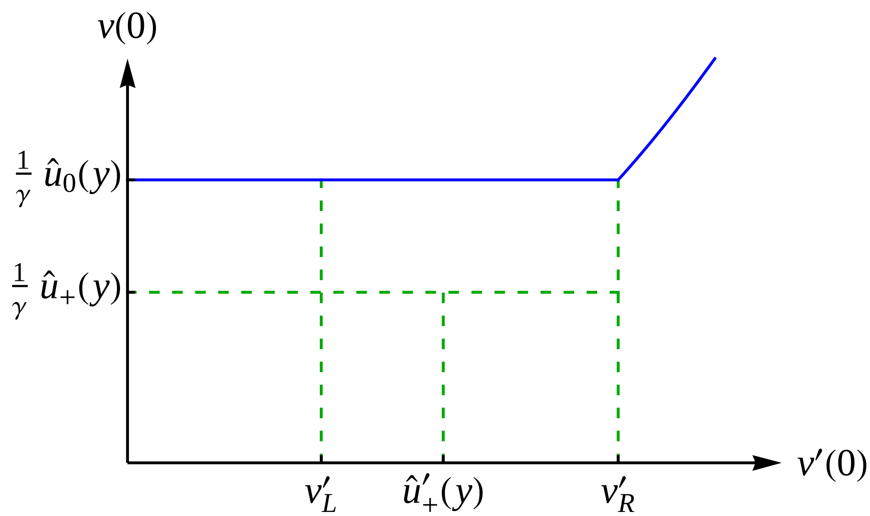
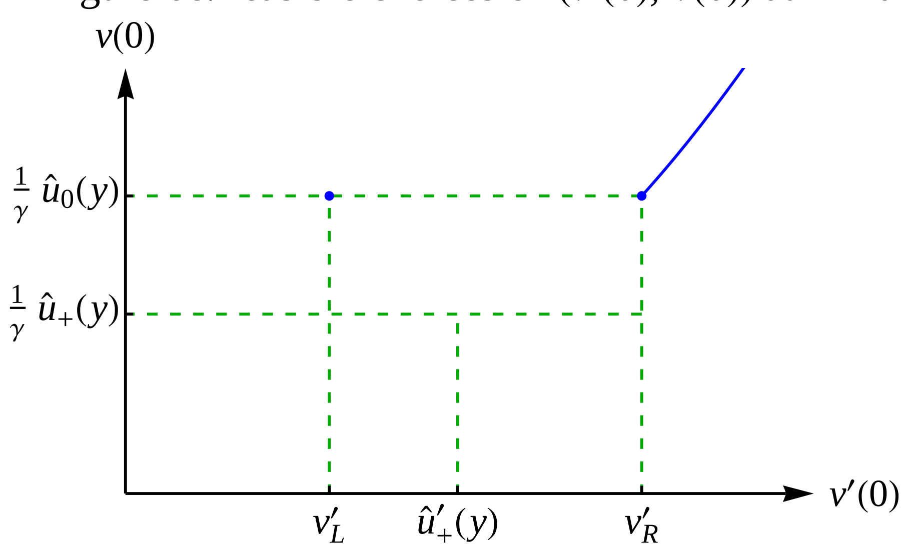


Figure 6c: feasible choices of $(v'(0), v(0))$ at $x = 0$



2. *there exists $x_0 > 0$ such that $v''(x_0) \geq 0$.*

Then $v''(x) > 0$ for all $x > x_0$.

Proof. Differentiating the Bellman equation of the $\widehat{u}$ consumer (i.e. equation (50)) with respect to x, we obtain

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v''' + ((\sigma^2 + \mu)\,x + y - \widehat{c})\,v'' - (\gamma - \mu)\,v'. \tag{53}$$

Now suppose that there exists $x_0 \geq 0$ such that $v''(x_0) > 0$, and suppose for a contradiction that there exists $x_1 > x_0$ such that $v''(x_1) \leq 0$. Let x_2 be the leftmost point in $(x_0, x_1]$ such that $v''(x_2) \leq 0$. Since $v'' > 0$ on $[x_0, x_2)$, we must actually have $v''(x_2) = 0$. Equation (53) then yields

$$v'''(x_2) = \frac{(\gamma - \mu)\,v'(x_2)}{\frac{1}{2}\,\sigma^2\,x_2^2}, \tag{54}$$

and the latter expression is strictly positive because $\gamma > \mu$ (by assumption), $v'(x_2) > 0$ (by Proposition 14) and $x_2 > 0$ (by construction). We therefore have $v'' < 0$ to the left of x_2, which is the required contradiction.

It remains to consider the case in which there exist $x_0 > 0$ such that $v''(x_0) = 0$. In that case (54) imples that $v'''(x_0) > 0$. But then there exists $\widetilde{x}_0 > x_0$ such that $v''(\widetilde{x}_0) > 0$. We are then back in the previous case. ■

Combining this result with the fact that v satisfies the Bellman equation of the $\widehat{u}$ consumer on the boundary of the wealth space, we obtain the following corollary.

Corollary 16. *Suppose that $\mu < \gamma$. Then $v'' < 0$ for all $x \geq 0$.*

In particular, $\widehat{c}' > 0$ on $(0, \infty)$. This is the case in which the $\widehat{u}$ consumer chooses the high-consumption strategy at all wealth levels.

Proof. Suppose for a contradiction that there exists $x_0 > 0$ such that $v''(x_0) \geq 0$. Then Proposition 15 implies that v is convex on $[x_0, \infty)$. This contradicts inequalities (49), which tell us that v is bounded above by the strictly concave function $\overline{v}$. We therefore have $v'' < 0$ for all $x > 0$. This in turn implies that $v''(0) \leq 0$. Now, letting $x \downarrow 0$ in equation (53) and rearranging, we obtain

$$v''(0) = \frac{(\gamma - \mu)\,v'(0)}{y - \widehat{c}(0+)}.$$

So: either $v'(0) = v'_L$, in which case $y - \widehat{c}(0+) < 0$ and therefore $v''(0) < 0$; or $v'(0) \geq v'_R$, in which case $y - \widehat{c}(0+) > 0$ and therefore $v''(0) > 0$. (Recall that $\gamma - \mu > 0$ by assumption.) Since $v''(0) \leq 0$, we must be in the first of these two cases. In particular, $v''(0) < 0$. This completes the proof. ■

Actually, the proof of Corollary 16 shows more:

Corollary 17. *Suppose that $\mu < \gamma$. Then $v'(0) = v'_L$.* ■

In other words, if $\mu < \gamma$ then the low-shadow-value boundary configuration obtains. In particular, $\widehat{c}(0+) = \psi\,\overline{c} > \widehat{c}(0) = y$.

G.5. Once concave, always strictly concave. The following result likewise only uses the fact that v satisfies the Bellman equation of the $\widehat{u}$ consumer in the interior of the wealth space.

Proposition 18. *Suppose that $\mu > \gamma$ and that either*

1. *there exists $x_0 \geq 0$ such that $v''(x_0) < 0$, or*
2. *there exists $x_0 > 0$ such that $v''(x_0) \leq 0$.*

Then $v''(x) < 0$ for all $x > x_0$.

Proof. The proof is completely analogous to that of Proposition 15. ■

Combining this result with the fact that v satisfies the Bellman equation of the $\widehat{u}$ consumer on the boundary of the wealth space, we obtain the following corollary.

Corollary 19. *Suppose that $\mu > \gamma$. Then either*

1. *there exists $\overline{x} \in (0, \infty)$ such that $v'' > 0$ on $(0, \overline{x})$, and $v'' < 0$ on $(\overline{x}, \infty)$; or*
2. *$v'' < 0$ for all $x \geq 0$.*

In particular: either there exists $\overline{x} \in (0, \infty)$ such that $\widehat{c}' < 0$ on $(0, \overline{x})$, and $\widehat{c}' > 0$ on $(\overline{x}, \infty)$; or $\widehat{c}' > 0$ on $[0, \infty)$. The first case is the case in which the $\widehat{u}$ consumer chooses the high-consumption strategy at low wealth levels and the low-consumption strategy at high wealth levels. The second case is the case in which the $\widehat{u}$ consumer chooses the low-consumption strategy at all wealth levels.[45]

Proof. Let X_0 be the set of all $x_0 \in [0, \infty)$ such that $v''(x_0) \leq 0$. Inequalities (49) imply that we cannot have $v'' > 0$ for all $x \geq 0$, so X_0 is non-empty. Let $\overline{x}$ be the smallest element of X_0. There are then two possibilities: either $\overline{x} > 0$ or $\overline{x} = 0$. If $\overline{x} > 0$, then $v'' > 0$ for all $x \in [0, \overline{x})$. Hence $v''(\overline{x}) = 0$, and Proposition 18 implies that $v'' < 0$ for all $x \in (\overline{x}, \infty)$. On the other hand, if $\overline{x} = 0$ then our construction of $\overline{x}$ yields only that $v''(0) \leq 0$. However, as in the proof of Corollary 16, we have

$$v''(0) = \frac{(\gamma - \mu)\, v'(0)}{y - \widehat{c}(0+)}.$$

Moreover: either $v'(0) = v'_L$, in which case $y - \widehat{c}(0+) < 0$ and therefore $v''(0) > 0$; or $v'(0) \geq v'_R$, in which case $y - \widehat{c}(0+) > 0$ and therefore $v''(0) < 0$. (Recall that we now have $\gamma - \mu < 0$.) Since $v''(0) \leq 0$, we must be in the second of these two cases. In particular, $v''(0) < 0$. We conclude that $v'' < 0$ for all $x \geq 0$ when $\overline{x} = 0$. ■

Actually, the proof of Corollary 19 shows slightly more:

Corollary 20. *Suppose that $\mu > \gamma$. Then either*

1. *$v'(0) = v'_L$, in which case $v''(0) > 0$; or*
2. *$v'(0) \geq v'_R$, in which case $v''(0) < 0$.* ■

In other words, if $\mu > \gamma$ then either the low-shadow-value boundary configuration occurs, in which case $\widehat{c}' < 0$ for small positive x, or the high-shadow-value boundary configuration occurs, in which case $\widehat{c}' > 0$ for all positive x. If the low-shadow-value boundary configuration obtains then $\widehat{c}(0+) = \psi\, \overline{c} > \widehat{c}(0) = y$. The main surprise in this case is the way in which: (i) the initial increase in consumption is confined to a single upward jump in $\widehat{c}$ at $x = 0$; and (ii) the decrease in consumption – as the consumer adjusts

[45] Notice that, in the first case, we have $\overline{x} > 0$ and $\widehat{c}'(\overline{x}) = 0$. One might therefore have expected to find a knife-edge case between the first and second cases in which $\overline{x} = 0$, $\widehat{c}'(0) = 0$ and $\widehat{c}'(x) > 0$ for all $x > 0$. This possibility is ruled out by Corollary 19. The reason why it does not arise is that $\widehat{c}'(0)$ does not vary continuously as the parameters of the model vary. Specifically, if $\overline{x} \downarrow 0$ as the parameter vector converges to an appropriate limit, then $\widehat{c}'(0)$ jumps up at the limit (and $v''(0)$ jumps down). To put the same point another way, away from the limit the low-shadow-value boundary configuration obtains, but at the limit the high-shadow-value boundary configuration obtains. Moreover it cannot happen that both boundary configurations obtain simultaneously.

to the low-consumption strategy – begins immediately to the right of $x = 0$. On the other hand, if the high-shadow-value boundary configuration obtains, then $\widehat{c}(0+) = \widehat{c}(0) \leq \psi\, y$.

G.6. The $\widehat{u}_+$ consumer. Combining Corollaries 16, 17, 19 and 20, we can tentatively identify three cases:

1. $\mu < \gamma$ and the low-shadow-value boundary configuration obtains.
2. $\mu > \gamma$ and the low-shadow-value boundary configuration obtains.
3. $\mu > \gamma$ and the high-shadow-value boundary configuration obtains.

However, we have not yet identified the borderline between cases 2 and 3. In order to locate this borderline, it will be helpful to consider a consumer who

- discounts the future exponentially at rate γ,
- faces the same wealth dynamics as the IG consumer and
- has the wealth-independent utility function $\widehat{u}_+$.

We call this consumer the $\widehat{u}_+$ consumer.

Let the value function of the $\widehat{u}_+$ consumer be $\widehat{v}_+ = \widehat{v}_+(x;\mu)$, where we have made explicit the dependence of $\widehat{v}_+$ on the parameter μ. Then:

Proposition 21.

1. *The low-shadow-value boundary configuration obtains if and only if* $\widehat{v}_+(0;\mu) < \frac{1}{\gamma}\,\widehat{u}_0(y)$.
2. *The high-shadow-value boundary configuration obtains if and only if* $\widehat{v}_+(0;\mu) \geq \frac{1}{\gamma}\,\widehat{u}_0(y)$. ■

The point here is that the $\widehat{u}$ consumer effectively has two options when $x = 0$: either exploit the utility premium available at $x = 0$ to the full, by consuming y and remaining at 0; or dispense with the utility premium altogether. The first option yields $\frac{1}{\gamma}\,\widehat{u}_0(y)$, and the second yields $\widehat{v}_+(0;\mu)$. If the first option is strictly better than the second, then the low-shadow-value boundary configuration obtains. If the second option is at least as good as the first, then the high-shadow-value boundary configuration obtains. It should also be noted that

$$v(0;\mu) = \max\{\tfrac{1}{\gamma}\,\widehat{u}_0(y), \widehat{v}_+(0;\mu)\},$$

where we have made explicit the dependence of v on the parameter μ.

Since $\widehat{v}_+$ is the value function of a standard optimization problem, we can use standard arguments to find those of its properties that are relevant to us. These properties are summarized in the following proposition.

Proposition 22.

1. *$\widehat{v}_+(0;\mu)$ is non-decreasing and continuous in μ for $\mu \in (-\infty, \overline{\mu})$.*

2. *$\widehat{v}_+(0;\mu) = \frac{1}{\gamma}\,\widehat{u}_+(y)$ for all $\mu \in (-\infty, \gamma]$.*

3. *$\widehat{v}_+(0;\mu)$ is strictly increasing in μ for $\mu \in [\gamma, \overline{\mu})$.*

4. *$\widehat{v}_+(0;\mu) \uparrow \frac{1}{\gamma}\,\widehat{u}_+(\infty)$ as $\mu \uparrow \overline{\mu}$.* ■

Noting that $\widehat{u}_+(y) < \widehat{u}_0(y) < \widehat{u}_+(\infty)$, we see that there is a unique $\mu_1 \in (\gamma, \overline{\mu})$ such that: (i) $\widehat{v}_+(0;\mu) < \frac{1}{\gamma}\,\widehat{u}_0(y)$ for $\mu < \mu_1$; (ii) $\widehat{v}_+(0;\mu_1) = \frac{1}{\gamma}\,\widehat{u}_0(y)$; and (iii) $\widehat{v}_+(0;\mu) > \frac{1}{\gamma}\,\widehat{u}_0(y)$ for $\mu > \mu_1$. The borderline between cases 2 and 3 therefore occurs at $\mu = \mu_1$.

G.7. From $\widehat{c}$ to c. At this point we have shown that there exists $\mu_1 \in (\gamma, \overline{\mu})$ such that:

1. If $\mu < \gamma$ then the low-shadow-value boundary configuration holds. I.e. $v(0) = \frac{1}{\gamma}\,\widehat{u}_0(y)$ and $v'(0) = v'_L = \widehat{u}'_+(\psi\,\overline{c}) < \widehat{u}'_+(y)$. This implies that $\widehat{c}(0+) = \psi\,\overline{c} > \widehat{c}(0) = y$. We also have: $v''(0) < 0$; and $\widehat{c}' > 0$ on $(0, \infty)$.

2. If $\gamma < \mu < \mu_1$ then the low-shadow-value boundary configuration still holds. I.e. we still have $v(0) = \frac{1}{\gamma}\,\widehat{u}_0(y)$ and $v'(0) = v'_L = \widehat{u}'_+(\psi\,\overline{c}) < \widehat{u}'_+(y)$. This implies that $\widehat{c}(0+) = \psi\,\overline{c} > \widehat{c}(0) = y$, as before. However, we now have: $v''(0) > 0$; and there exists $\overline{x} \in (0, \infty)$ such that $\widehat{c}' < 0$ on $(0, \overline{x})$ and $\widehat{c}' > 0$ on $(\overline{x}, \infty)$.

3. If $\mu > \mu_1$ then the high-shadow-value boundary configuration holds, i.e. $v(0) > \frac{1}{\gamma}\,\widehat{u}_0(y)$ and $v'(0) > v'_R = \widehat{u}'_+(\psi\,y) > \widehat{u}'_+(y)$. This implies that $\widehat{c}(0+) = \widehat{c}(0) < \psi\,y$. We also have: $v''(0) < 0$; and $\widehat{c}' > 0$ on $(0, \infty)$.

In order to deduce the behaviour of c in these three cases, note that:

- For $x > 0$, c is determined by $u'(c) = \beta\, v'$ and $\widehat{c}$ is determined by $\widehat{u}'_+(\widehat{c}) = v'$. Also, the formula for $\widehat{u}_+$ given in the proof of Theorem 4 implies that

$$\widehat{u}'_+(\widehat{c}) = \tfrac{1}{\beta}\, u'(\tfrac{1}{\psi}\, \widehat{c}). \tag{55}$$

  Hence $u'(c) = \beta\, v' = \beta\, \widehat{u}'_+(\widehat{c}) = u'(\frac{1}{\psi}\, \widehat{c})$. Hence $c = \frac{1}{\psi}\, \widehat{c}$.

- For $x = 0$, c is determined by $u'(c) = \max\{u'(y), \beta\, v'\}$ and $\widehat{c}$ is determined by $\widehat{c} \in \operatorname{argmax}_{\widehat{c}\in(0,y]}\{\widehat{u}_0(\widehat{c}) - v'\,\widehat{c}\}$. Now $\beta\, v' > u'(y)$ iff $v' > \widehat{u}'_+(\psi\, y)$, because $\widehat{u}'_+(\psi\, y) = \frac{1}{\beta}\, u'(y)$ by (55). And, in this case, $u'(c) = \beta\, v'$ and $\widehat{u}'_0(\widehat{c}) = \widehat{u}'_+(\widehat{c}) = v'$. Hence $c = \frac{1}{\psi}\, \widehat{c}$. Similarly, $\beta\, v' < u'(y)$ iff $v' < \widehat{u}'_+(\psi\, y)$. However, in this case, $c = \widehat{c} = y$.

- Provided that $\mu \neq \mu_1$, we have either

$$v'(0) = v'_L = \widehat{u}'_+(\psi\, \overline{c}) = \tfrac{1}{\beta}\, u'(\overline{c}) < \tfrac{1}{\beta}\, u'(y)$$

  or

$$v'(0) > v'_R = \widehat{u}'_+(\psi\, y) = \tfrac{1}{\beta}\, u'(y).$$

  In particular, the case $\beta\, v'(0) = u'(y)$ does not arise.

Combining these observations with points 1-3 above, we conclude that:

1. If $\mu < \gamma$ then $c(0+) = \frac{1}{\psi}\, \widehat{c}(0+) = \overline{c} > y = c(0)$ and $c' = \frac{1}{\psi}\, \widehat{c}' > 0$ on $(0, \infty)$.
2. If $\gamma < \mu < \mu_1$ then we still have $c(0+) = \frac{1}{\psi}\, \widehat{c}(0+) = \overline{c} > y = c(0)$. But now $c' = \frac{1}{\psi}\, \widehat{c}' < 0$ on $(0, \overline{x})$ and $c' = \frac{1}{\psi}\, \widehat{c}' > 0$ on $(\overline{x}, \infty)$.
3. If $\mu > \mu_1$ then $c(0+) = \frac{1}{\psi}\, \widehat{c}(0+) = \frac{1}{\psi}\, \widehat{c}(0) < y$. We also have $c' = \frac{1}{\psi}\, \widehat{c}' > 0$ on $(0, \infty)$.

The point is that, in the three cases (i) $x > 0$, (ii) $x \downarrow 0$ and (iii) $x = 0$ and the high-shadow-value boundary configuration obtains, the behaviour of c can be deduced from that of $\widehat{c}$ via the simple formula $c = \frac{1}{\psi}\, \widehat{c}$. (Since $\psi < 1$, this formula captures the idea that the IG consumer will overconsume compared with the $\widehat{u}$ consumer.) And, when $x = 0$ and the low-shadow-value boundary configuration obtains, we have $c = \widehat{c} = y$. This completes the proof of Theorems 9, 11 and 10.

G.8. The borderline cases $\mu = \gamma$ and $\mu = \mu_1$. Up to now we have said relatively little about the borderline cases. The case $\mu = \gamma$ has several interesting features. First, letting $\mu \uparrow \gamma$, we see that $v'' \leq 0$ and $\widehat{c}' \geq 0$ on $[0, \infty)$. Second, again letting $\mu \uparrow \gamma$, we obtain $\widehat{v}_+(0) = \frac{1}{\gamma}\,\widehat{u}_+(y) < \frac{1}{\gamma}\,\widehat{u}_0(y)$. It follows that the low-shadow-value boundary configuration obtains. This in turn implies that $\widehat{c}(0+) = \psi\,\overline{c} > y$. Letting $x \downarrow 0$ in equation (53) then yields $0 = (y - \widehat{c}(0+))\,v''(0)$. It follows that $v''(0) = 0$. Third, by considering higher-order analogues of equation (53), one can go on to show that $v^{(n)}(0) = 0$ for all $n \geq 3$ as well. In other words, the only non-zero coefficients in the Taylor expansion for v at $x = 0$ are $v(0)$ and $v'(0)$. At first sight this would seem to suggest that v is linear. However, this would contradict inequalities (49). The resolution lies in the fact that, while v is *smooth* at 0, it is *not analytic* at 0. Rather, v' (and therefore $\widehat{c}$) are so called 'flat functions'. (In other words, they are smooth functions, all the Taylor coefficients of which vanish except the first.) This terminology turns out to be apt: simulations show that v' and $\widehat{c}$ are nearly constant for a significant interval of wealth starting at $x = 0$.

The case $\mu = \mu_1$ involves a number of subtleties. First, even though μ_1 is the point at which we switch from the left- to the high-shadow-value boundary configuration, only the high-shadow-value boundary configuration can occur when $\mu = \mu_1$. This is because $v'(0)$ is essentially the limit $v'(0+)$, and as such is determined by behavior in the interior of the wealth space. Moreover, in the interior of the wealth space, the low-consumption strategy is the preferred strategy of the $\widehat{u}$ consumer. The $\widehat{u}$ consumer does, however, have two equally good options at $x = 0$: since $v'(0) = \widehat{u}'_+(\psi\,y)$ and $\widehat{u}_0$ has slope $\widehat{u}'_+(\psi\,y)$ on $[\psi\,y, y]$, she is indifferent between $\widehat{c}(0) = \psi\,y$ and $\widehat{c}(0) = y$. (She is in fact indifferent among all $\widehat{c}(0) \in [\psi\,y, y]$, but the intermediate options should be seen as the result of strictly randomizing between $\psi\,y$ and y. Moreover they all lead to the same outcome as $\psi\,y$: the dynamics move immediately into the interior of the wealth space.) If she chooses $\widehat{c}(0) = \psi\,y$, then she embarks immediately on the low-consumption strategy. If she chooses $\widehat{c}(0) = y$, then she remains forever with wealth 0. Either way, she ends up with the payoff $v(0) = \frac{1}{\gamma}\,\widehat{u}_0(y)$. Second, as $\mu \uparrow \mu_1$, the length of the interval over which $\widehat{c}$ decreases – which is always an open interval with left-hand endpoint 0 – converges to 0. (So, in effect, $\widehat{c}$ jumps up from y to $\psi\,\overline{c}$ at 0 and then decreases very rapidly back down to something close to $\psi\,y$.) In other words, a boundary layer develops near $x = 0$.

H. GENERALIZING THE UTILITY FUNCTION

The core results of our paper, namely the Value-Function Equivalence Theorem (Theorem 4) and the Existence and Uniqueness Theorem (Theorem 5), can be proved under assumptions much weaker than A1-A3. The purpose of this appendix is to describe these weaker assumptions, and to explain briefly why Theorems 4 and 5 continue hold under them. For a more detailed discussion, see the 2011 draft of our paper (Harris and Laibson (2011): "Instantaneous Gratification," Harvard University mimeo).

H.1. The Weaker Assumptions. Our weaker assumptions are formulated in terms of the (non-constant) relative risk aversion and the (non-constant) relative prudence (Kimball 1990) of u, namely

$$\rho(c) \equiv -\frac{c\,u''(c)}{u'(c)} \quad \text{and} \quad \pi(c) \equiv -\frac{c\,u'''(c)}{u''(c)}.$$

Notice that both ρ and π are functions: for each consumption level c, they tell us the coefficient of relative risk aversion and the coefficient of relative prudence at c. The weaker assumptions are:

B1 There exist constants $\underline{\rho}, \overline{\rho} \in (0, \infty)$ such that $\underline{\rho} \leq \rho(c) \leq \overline{\rho}$ and $\underline{\rho} + 1 \leq \pi(c) \leq \overline{\rho} + 1$ for all $c \in (0, \infty)$.

B2 $1 - \beta < \dfrac{\underline{\rho}}{1 + \overline{\rho} - \underline{\rho}}$.

B3 $\mu < \dfrac{1}{1 - \underline{\rho}}\,\gamma + \frac{1}{2}\,\underline{\rho}\,\sigma^2$ if $\underline{\rho} < 1$.

Assumption B1 is less restrictive than it might appear at first sight. Indeed, if $0 < \underline{\rho} \leq \rho \leq \overline{\rho} < \infty$ and $1 < \underline{\pi} \leq \pi \leq \overline{\pi} < \infty$, then it can be shown that in fact $\underline{\rho} \geq \underline{\pi} - 1$ and $\overline{\rho} \leq \overline{\pi} - 1$. In other words, as long as we know that relative risk aversion is bounded at all, then we know that it is bounded in terms of the bounds on relative prudence. There is therefore no loss of generality in putting $\underline{\rho} = \underline{\pi} - 1$ and $\overline{\rho} = \overline{\pi} - 1$ or, as we have done here, $\underline{\pi} = \underline{\rho} + 1$ and $\overline{\pi} = \overline{\rho} + 1$. Assumption B2 requires that the dynamic inconsistency of the IG consumer (as measured by $1 - \beta$) must be smaller: (i) the lower the minimum possible coefficient of relative risk aversion (as measured by $\underline{\rho}$); and (ii) the larger the fluctuations in the coefficients of relative risk aversion and relative prudence (as measured by $\overline{\rho} - \underline{\rho}$). Assumption B3 requires that the expected return on the financial asset (as measured by μ) be smaller, the lower the minimum possible coefficient of relative risk aversion (as measured by $\underline{\rho}$). This is because, the lower $\underline{\rho}$, the faster the potential growth of u.

Assumptions B1-B3 are much weaker than Assumptions A1-A3, and they are only marginally more complicated to state. (That they are strictly weaker follows form the fact that they reduce to Assumptions A1-A3 in the CRRA case, namely the case $\overline{\rho} = \underline{\rho}$.)

H.2. Extending the Analysis. Under Assumptions B1-B3, we have the following generalization of the Value-Function Equivalence Theorem (Theorem 4).

Theorem 23 [Generalization of Value-Function Equivalence]**.** *Theorem 4 holds as stated under the weaker Assumptions B1-B3.*

The key point of the proof is to show that, even under the weaker Assumptions B1-B3, we can still find a wealth-dependent utility function $\widehat{u}$ such that the Bellman equation of the IG consumer is identical to the Bellman equation of the $\widehat{u}$ consumer.

Proof. The first step is to construct $\widehat{u}_+ : (0, \infty) \to \mathbb{R}$. As in the proof of Theorem 4, we begin by defining a function $h_+ : (0, \infty) \to \mathbb{R}$ by the formula

$$h_+(\alpha) = u(f_+(\beta\,\alpha)) - \alpha\, f_+(\beta\,\alpha),$$

where $f_+(\alpha)$ is the unique c satisfying $u'(c) = \alpha$. Assumptions B1-B2 then imply:[46]

H1 $h'_+(\alpha) < 0$ and $h''_+(\alpha) > 0$ for all $\alpha > 0$; and

H2 there exist $0 < \underline{\theta} \leq \overline{\theta} < \infty$ such that $\underline{\theta} \leq -\dfrac{\alpha\, h''_+(\alpha)}{h'_+(\alpha)} \leq \overline{\theta}$ for all $\alpha > 0$.

Next, we define a function $\widehat{u}_+ : (0, \infty) \to \mathbb{R}$ by the formula

$$\widehat{u}_+(\widehat{c}) = \min_{\alpha \in (0,\infty)} h_+(\alpha) + \widehat{c}\,\alpha.$$

It can be verified that

U1 $\widehat{u}'_+(\widehat{c}) > 0$ and $\widehat{u}''_+(\widehat{c}) < 0$ for all $\widehat{c} > 0$; and

U2 $\overline{\theta}^{-1} \leq -\dfrac{\widehat{c}\,\widehat{u}''_+(\widehat{c})}{\widehat{u}'_+(\widehat{c})} \leq \underline{\theta}^{-1}$ for all $\widehat{c} > 0$.

[46] For a brief explanation of why Assumptions B1-B2 imply H1-H2, see Section H.3 below.

Finally, we define a function $\widehat{h}_+ : (0, \infty) \to \mathbb{R}$ by the formula

$$\widehat{h}_+(\alpha) = \max_{\widehat{c} \in (0,\infty)} \widehat{u}_+(\widehat{c}) - \alpha\, \widehat{c}$$

for all $\alpha > 0$.

The second step is to construct $\widehat{u}_0 : (0, y] \to \mathbb{R}$. We begin by defining a function $h_0 : (-\infty, \infty) \to \mathbb{R}$ by the formula

$$h_0(\alpha) = u(f_0(\beta\, \alpha)) - \alpha\, f_0(\beta\, \alpha),$$

where $f_0(\alpha)$ is the unique c satisfying $u'(c) = \max\{u'(y), \alpha\}$. It is easy to check that

$$h_0(\alpha) = \left\{ \begin{array}{ll} u(y) - \alpha\, y & \text{for } \alpha \in (-\infty, \frac{1}{\beta}\, u'(y)] \\ h_+(\alpha) & \text{for } \alpha \in [\frac{1}{\beta}\, u'(y), \infty) \end{array} \right\}.$$

Next, we define a function $\widehat{u}_0 : (0, y] \to \mathbb{R}$ by the formula

$$\widehat{u}_0(\widehat{c}) = \min_{\alpha \in (-\infty,\infty)} h_0(\alpha) + \widehat{c}\, \alpha.$$

It can be verified that

$$\widehat{u}_0(\widehat{c}) = \left\{ \begin{array}{ll} \widehat{u}_+(\widehat{c}) & \text{for } \widehat{c} \in (0, \psi(y)\, y] \\ \widehat{u}_+(\psi(y)\, y) + (\widehat{c} - \psi(y)\, y)\, \widehat{u}_+'(\psi(y)\, y) & \text{for } \widehat{c} \in [\psi(y)\, y, y] \end{array} \right\},$$

where

$$\psi(y) = \frac{\rho(y) - (1 - \beta)}{\rho(y)}$$

and $\rho(y)$ is the relative risk aversion of u at y. Finally, we define a function $\widehat{h}_0 : (-\infty, \infty) \to \mathbb{R}$ by the formula

$$\widehat{h}_0(\alpha) = \max_{\widehat{c} \in (0,y]} \widehat{u}_0(\widehat{c}) - \alpha\, \widehat{c}$$

for all $\alpha \in (-\infty, \infty)$.

The third and final step is to note that, because h_+ and h_0 are both convex, it follows that $\widehat{h}_+ = h_+$ and $\widehat{h}_0 = h_0$. This is an instance of convex duality, the basic reference for which is Rockafellar (1970). In particular, the Bellman equation of the IG consumer and the Bellman equation of the $\widehat{u}$ consumer coincide. ■

Armed with Theorem 23, we can immediately deduce the following theorem using the same arguments that we used to deduce Theorem 5 from Theorem 4.

Theorem 24 [Generalization of Existence and Uniqueness]**.** *Theorem 5 holds as stated under the weaker Assumptions B1-B3.* ■

H.3. Proof that Assumptions B1-B2 Imply Assumptions H1-H2. It can be verified by direct calculation that

$$h'_+ = -\frac{(\rho(f_+) - (1-\beta))\, f_+}{\rho(f_+)}$$

and

$$h''_+ = -\frac{\beta}{u''(f_+)\,\rho(f_+)}\left((2-\beta)\,\rho(f_+) - (1-\beta)\,\pi(f_+)\right),$$

where $h_+ = h_+(\alpha)$ and $f_+ = f_+(\beta\,\alpha)$. Hence

$$-\frac{\alpha\, h''_+}{h'_+} = \frac{(2-\beta)\,\rho(f_+) - (1-\beta)\,\pi(f_+)}{(\rho(f_+) - (1-\beta))\,\rho(f_+)},$$

where we have used the fact that

$$-\frac{\alpha\,\beta}{u''(f_+)\, f_+} = -\frac{u'(f_+)}{u''(f_+)\, f_+} = \frac{1}{\rho(f_+)}.$$

Now, considering the numerator in the expression for h'_+, we have

$$\rho(f_+) - (1-\beta) \geq \underline{\rho} - (1-\beta) \geq \frac{\underline{\rho}}{1+\overline{\rho}-\underline{\rho}} - (1-\beta) > 0,$$

where the first inequality follows from Assumption B1, the second from the fact that $\overline{\rho} - \underline{\rho} \geq 0$ and the third from Assumption B2. Hence $h'_+ < 0$. Similarly, considering the term in parentheses in the expression for h''_+, we have

$$\begin{aligned}
(2-\beta)\,\rho(f_+) - (1-\beta)\,\pi(f_+) &\geq (2-\beta)\,\underline{\rho} - (1-\beta)\,(\overline{\rho}+1) \\
&= \underline{\rho} - (1-\beta)\,(1+\overline{\rho}-\underline{\rho}) \\
&> 0
\end{aligned}$$

where the first relation follows from Assumption B1 and the third from Assumption B2. Hence $h''_+ < 0$. Finally, again considering the numerator in the expression for h'_+ and the term in parentheses in the expression for h''_+, we have

$$\rho(f_+) - (1-\beta) \leq \overline{\rho} - (1-\beta)$$

and

$$(2-\beta)\,\rho(f_+) - (1-\beta)\,\pi(f_+) \leq (2-\beta)\,\overline{\rho} - (1-\beta)\,(\underline{\rho}+1).$$

Hence

$$\underline{\theta} \leq -\frac{\alpha\, h''_+(\alpha)}{h'_+(\alpha)} \leq \overline{\theta},$$

where

$$\underline{\theta} = \frac{(2-\beta)\,\underline{\rho} - (1-\beta)\,(\overline{\rho}+1)}{\overline{\rho} - (1-\beta)} \quad \text{and} \quad \overline{\theta} = \frac{(2-\beta)\,\overline{\rho} - (1-\beta)\,(\underline{\rho}+1)}{\underline{\rho} - (1-\beta)}.$$