Dynamic programming with time inconsistency in continuous time

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Outline for this lecture:

1. Summary of continuous time methods (stochastic calculus).
2. Continuous time implementation
3. From discrete time to continuous time

1 Summary of continuous-time methods:

Definition: If a continuous time stochastic process, $z(t)$, is **Brownian Motion**, then $z(t') - z(t)$ satisfies the following conditions:

1. $z(t') - z(t) \sim N(0, t' - t)$
2. If $t \leq t' \leq t'' \leq t'''$,

$$E\left[(z(t') - z(t))(z(t''') - z(t''))\right] = 0.$$

Definition: An Ito Process has instantaneous drift $a(x, t)$ and instantaneous standard deviation $b(x, t)$. We adopt the notational convention of writing this as

$$dx = a(x, t)dt + b(x, t)dz,$$

where dz are Brownian increments.

Theorem 1.1 (Ito's Lemma) *Let $z(t)$ be a Wiener Process (i.e. Brownian Motion). Let $x(t)$ be an Ito Process with $dx = a(x,t)dt + b(x,t)dz$. Let $v = v(x,t)$ be twice differentiable, then*

$$\begin{aligned} dv &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}dx + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}b(x,t)^2 dt. \\ E[dv] &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}a(x,t)dt + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}b(x,t)^2 dt. \end{aligned}$$

Deriving a continuous-time Bellman Equation:

Let $U(x, u, t) =$ instantaneous payoff function, where x is state variable, u is control variable and t is time. Let $x' = x + \Delta x$ and $t' = t + \Delta t$. So

$$v(x, t) = \max_u \left\{ U(x, u, t)\Delta t + (1 + \gamma \Delta t)^{-1} E v(x', t') \right\}$$

$$(1 + \gamma \Delta t) v(x, t) = \max_u \left\{ (1 + \gamma \Delta t) U(x, u, t)\Delta t + E v(x', t') \right\}$$

$$\gamma v(x, t)\Delta t = \max_u \left\{ (1 + \gamma \Delta t) U(x, u, t)\Delta t + E v(x', t') - v(x, t) \right\}$$

Multiply out and let $\Delta t \rightarrow 0$. Terms of order $(dt)^2 = 0$.

$$\gamma v(x, t) dt = \max_u \{ U(x, u, t) dt + E(dv) \} \qquad (*)$$

Now substitute in for $E(dv)$, using Ito's Lemma:

$$dv = \left[\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}a + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}b^2\right] dt + \frac{\partial v}{\partial x}bdz$$

where $a = a(x, u, t)$, $b = b(x, u, t)$, and $dx = a(x, u, t)dt + b(x, u, t)dz$. Since, $E\frac{\partial v}{\partial x}bdz = 0$, we have,

$$E(dv) = \left[\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}a + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}b^2\right] dt$$

Substituting this expression into equation (*), we get

$$\gamma v(x, t)dt = \max_u \left\{U(x, u, t)dt + \left[\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}a + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}b^2\right] dt\right\}$$

which is a partial differential equation (in x and t).

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2 Present bias in continuous time (Harris and Laibson 2013)

Transitions between (countable) selves occur at hazard rate λ.

Let τ_n be the (stochastic) duration of self n.

Each self values her next self discretely less than her present self, discounting it by the factor $0 < \beta \leq 1$.

All selves also discount exponentially with discount factor $0 < \delta < 1$. Let $\gamma = -\ln \delta$.

Discount function of self n.

$$D_n(t) = \left\{ \begin{array}{lll} \delta^t & \text{if} & t \in [0, \tau_n) \\ \beta\,\delta^t & \text{if} & t \in [\tau_n, \infty) \end{array} \right\}. \tag{1}$$

2.1 Application to a savings problem:

- Let c be consumption.
- Let y be a flow of labor income (non-collatoralizable).
- Let z be Brownian motion (defined on the next slide).
- Let x be cash on hand: $dx = (\mu x + y - c)dt + \sigma x dz$
- So dynamics for x are characterized by geometric brownian motion

- Constraints:

$$c \text{ unbounded if } x > 0,$$
$$c \leq y \text{ if } x = 0.$$

- Current value function, $w(x)$:

$$\gamma\, w = u(c) + (\mu\, x + y - c)\, w' + \tfrac{1}{2}\, \sigma^2\, x^2\, w'' + \lambda\, (\beta\, v - w) \tag{2}$$

- Continuation value function, $v(x)$:

$$\gamma\, v = u(c) + (\mu\, x + y - c)\, v' + \tfrac{1}{2}\, \sigma^2\, x^2\, v'' \tag{3}$$

- Consumption:

$$u'(c) = w'(x). \tag{4}$$

2.2 The Instantaneous Gratification Model: $\lambda \to \infty$.

- First differential equation collapses to:

$$w = \beta\, v \tag{5}$$

- Continuation value function, $v(x)$:

$$\gamma\, v = u(c) + (\mu\, x + y - c)\, v' + \tfrac{1}{2}\, \sigma^2\, x^2\, v'' \tag{6}$$

- Consumption:

$$u'(c) = w'(x) = \beta v'(x) \tag{7}$$

- In essence, three-equation system collapses to two-equation system (w is superfluous).

2.3 Properties of the IG model:

- Unique equilibrium (Bellman-system equivalence to an optimization problem; so policy functions are not equivalent).

- Because of value-function equivalence, Bellman system is numerically tractable (not a game).

- Consumption drops discontinuously when agents hit the liquidity constraint.

- Policy functions are smooth in the interior of the state space.

- Single welfare criterion. Recall that $w = \beta\, v$.

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3 From discrete time to continuous time (Laibson and Maxted 2022)

- In a calibrated dynamic consumption model, the desirable characteristics of continuous time models are obtained when time periods are at the weekly frequency (or any higher frequencies).

- Use "noise" in the discrete-time dynamic budget constraint that converges to Brownian motion as the time steps (Δ) get shorter:

$$x_{t+\Delta} = R^{\Delta}(x_t - c_t) + \sqrt{\Delta}\varepsilon_{t+\Delta},$$

where $\varepsilon_{t+\Delta}$ is uncorrelated noise with variance equal to the "annual" variance of shocks.

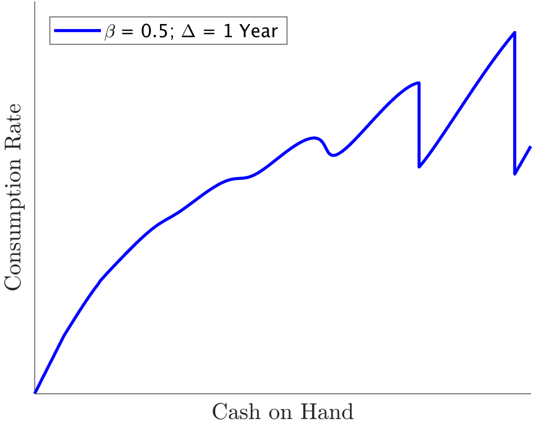
- Variance of the sum of shocks over a calender period of time:

$$\frac{1}{\Delta}V\left[\sqrt{\Delta}\varepsilon_{t+\Delta}\right] = V\left[\varepsilon\right].$$

To calibrate the model, set $V[\varepsilon]$ equal to the annualized level of noise.

Paper in One Slide

- Buffer-Stock Consumption-Savings Model:
  - Stochastic Income
  - Liquidity Constraint: Consumption $\leq$ Cash on Hand
  - Time-Step Δ Years



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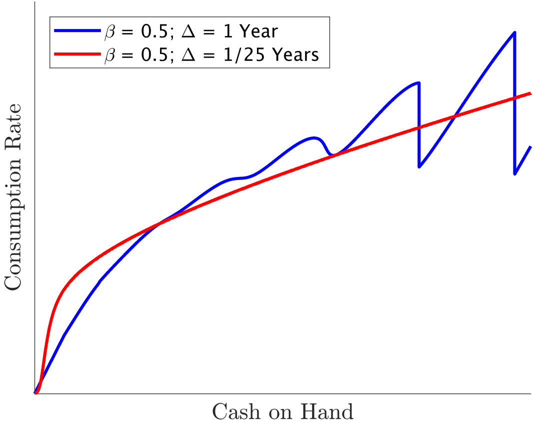

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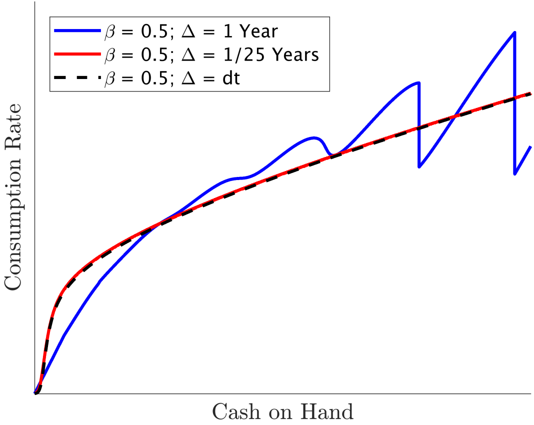

Two key lessons from Laibson and Maxted (2022) for modeling and numerical impelementation:

1. You can work in continuous time, which closely approximates time steps of an hour, a day, a week, OR

2. You can work in discrete time, with time steps of a week, which closely approximates much smaller time steps, even vanishing small time steps (see Augenblick 2018).

We also know that pathologies can be removed by either assuming perfect naivite or making β close to one (see Harris and Laibson 2003).

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