Dynamic programming with time inconsistency in discrete time

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Outline for this lecture:

1. Overview of Dynamic Programming with Present Bias
2. Dynamic Programming: Search
3. Dynamic Programming: Consumption

Next lecture: Continuous time

1 Overview: dynamic programming

Break the problem into two value functions: W and V. W is the value function that discounts with present bias (i.e., the "current" value function), so

$$W_t = u_t + \beta\delta V_{t+1}.$$

V is the value function that discounts exponentially (so V is the continuation value function).

$$V_t = u_t + \delta V_{t+1}.$$

Note that

$$\begin{aligned}
V_t &= u_t + \delta u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \ldots \\
W_t &= u_t + \beta\delta u_{t+1} + \beta\delta^2 u_{t+2} + \beta\delta^3 u_{t+3} + \ldots \\
W_t &= u_t + \beta\delta \left[u_{t+1} + \delta u_{t+2} + \delta^2 u_{t+3} + \ldots \right] \\
W_t &= u_t + \beta\delta V_{t+1}
\end{aligned}$$

Let a be an action. Let x be a point in the state space. Let $x_{t+1} = \Gamma(x_t, a_t)$ represent state dynamics.

Agent maximizes W, taking V as a *given* function.

$$a_t^* = \arg\max \left[u_t(a_t) + \beta\delta V_{t+1}(x_{t+1} = \Gamma(x_t, a_t)\right] \tag{1}$$

V is calculated using the equilibrium policies generated by the maximization of W.

$$V_t(x_t) = u_t(a_t^*) + \delta V_{t+1}(x_{t+1} = \Gamma(x_t, a_t^*)) \tag{2}$$

These two equations characterize equilibrium and they represent a fixed point problem. You need V to define the equilibrium policies. You need the equilibrium policies to define V.

This fixed point problem may have more than one solution.

2 Dynamic Programming: Search and Procrastination

- See Akerlof (1992) and O'Donoghue and Rabin (1999a,1999b) for early papers on Procrastination

- Today: Carroll, Choi, Laibson, Madrian, and Metrick (2009)

- $0 < \beta < 1$; $\delta = 1$ (interpretation: daily model so $\delta \simeq 1$)

- Per period loss from delay L (e.g., lost per-period benefit)

- Stochastic action cost c_t drawn from a uniform distribution on the interval $[\underline{c}, \bar{c}]$

2.1 Sophisticates

Let W represent the current cost function

$$W(c) = \begin{cases} c & \text{if act} \\ \beta\left[L + EV(c')\right] & \text{if wait} \end{cases} \tag{3}$$

Let V represent the "exponentially discounted" continuation cost function

$$V(c) = \begin{cases} c & \text{if act tomorrow} \\ L + EV(c') & \text{if wait tomorrow} \end{cases} \tag{4}$$

Study stationary equilibrium: "cutoff threshold" c^*.

Agents must be indifferent in the current period between acting and waiting at the cutoff,

$$c^* = \beta \left[L + EV(c') \right]. \tag{5}$$

Our problem can be reduced to two equations

$$\begin{aligned} c^* &= \beta [L + EV] \\ EV &= \int_{c \leq c^*} c dF(c) + \int_{c > c^*} [L + EV] \, dF(c) \end{aligned}$$

and two unknowns: c^* and EV.

Proposition 2.1 *The equilibrium cutoff threshold is*

$$c^* = \frac{\underline{c} + \sqrt{\underline{c}^2 \left[1 - (2-\beta)\beta\right] + 4\beta \left(1 - \frac{\beta}{2}\right)\left(\bar{c} - \underline{c}\right) L}}{2 - \beta}. \tag{6}$$

2.2 Properties of c^*:

- How does c^* change with L, the flow cost?

- How does c^* change with β, the short-term discount factor?

- Is c^* above $\underline{c}$?

- Is c^* below $\bar{c}$?

2.3 Procrastination with sophisticates

- Let c^{**} be the *desired* future threshold. So,

$$c^{**} \equiv c^{*}_{\beta=1} = \underline{c} + \sqrt{2\left(\bar{c} - \underline{c}\right) L}. \tag{7}$$

- How does c^{**} compare with $c^{*}_{\beta<1}$?

- What is the probability that an agent procrastinates in a given period?

- Illustrative calibration: $\underline{c} = 0$ and $\bar{c} = 1$.

$$c^{**} - c^{*} = \sqrt{2L}\left(1 - \sqrt{\frac{\beta/2}{1-\beta/2}}\right)$$

- If $\sqrt{2L} = 1$, exponentials always do it. If, $\beta = 2/3$ probability of procrastination is

$$c^{**} - c^{*} = 1 - \sqrt{\frac{\beta/2}{1 - \beta/2}} = 1 - \frac{1}{\sqrt{2}} = 0.29.$$

2.4 Testing your intuition:

- Consider the boundary case in which $\bar{c} = \underline{c}$.

- What is the equilibrium action rule?

2.5 Procrastination with naives:

- Let c_N^* be the equilibrium Naive threshold.

- Is c_N^* greater than or less than c^*?

- Is c_N^* greater than or less than $\underline{c}$?

- Is that the good news or the bad news for this model?

3 Dynamic programming: stationary consumption problem

- Let c represent consumption
- Let x represent cash-on-hand
- Let $\tilde{y}$ represent iid stochastic income
- Let R represent gross interest rate

$$x_{t+1} = R(x_t - c_t) + \tilde{y}_{t+1}$$

- A (Markov) strategy is a map from state x to control c.

- Let V be the continuation-value function, W be the current-value function and C be the consumption function. Then:

$$\begin{aligned} V(x) &= U(C(x)) + \delta\, \mathsf{E}[V(R(x - C(x)) + y)] \\ W(x) &= U(C(x)) + \beta\delta\, \mathsf{E}[V(R(x - C(x)) + y)] \\ C(x) &= \operatorname*{argmax}_{c} U(c) + \beta\delta\, \mathsf{E}[V(R(x - c) + y)] \end{aligned}$$

- Envelope Theorem: $W'(x) = U'(C(x))$.

- First-order-condition: $U'(C(x)) = R\beta\delta\, \mathsf{E}[V'(R(x - C(x)) + y)]$.

- Identity linking V and W: $\beta V(x) = W(x) - (1 - \beta)U(C(x))$.

3.1 Problem is recursive

- Start with V.

- Find C:

$$C(x) = \operatorname*{argmax}_{c} U(c) + \beta\delta\,\mathsf{E}[V(R(x-c)+y)]\,.$$

- Find $\hat{V}$:

$$\hat{V}(x) = U(C(x)) + \delta\,\mathsf{E}[V(R(x - C(x)) + y)]$$

- In this way, generate an operator $T : V \mapsto \hat{V}$.

3.2 Generalized Euler Equation (Harris and Laibson 2001).

We have

$$\begin{aligned} u'(c_t) &= R\beta\delta\, \mathsf{E}_t\Big[V'(x_{t+1})\Big] \\ &= R\delta\, \mathsf{E}_t\left[W'(x_{t+1}) - (1-\beta)u'(c_{t+1})\frac{dC_{t+1}}{dx_{t+1}}\right] \\ &= R\delta\, \mathsf{E}_t\left[u'(c_{t+1}) - (1-\beta)u'(c_{t+1})\frac{dC_{t+1}}{dx_{t+1}}\right]. \end{aligned}$$

Follows from FOC, differentiated identity, and envelope theorem. Simplifying,

$$u'(c_t) = R\, \mathsf{E}_t\left[\beta\delta\left(\frac{dC_{t+1}}{dX_{t+1}}\right) + \delta\left(1 - \frac{dC_{t+1}}{dX_{t+1}}\right)\right] u'(c_{t+1}).$$

Quasi-hyperbolics are highly patient when $\frac{dC_{t+1}}{dX_{t+1}} \simeq 0$, and highly impatient when $\frac{dC_{t+1}}{dX_{t+1}} \simeq 1$.

Calibration of steady state with no growth:

- $u(c) = \ln(c)$.

- In a standard exponential discounting model (i.e., $\beta = 1$), we have

$$\delta R = 1,$$

  so the discount rate, $-\ln \delta$, is equal to the interest rate, $\ln R$.

- What happens in the quasi-hyperbolic economy?

- Suppose $\beta = 2/3$ and $\delta = 0.99$, what is the steady state interest rate?

- If λ is the APC=MPC, then in steady state $(c_t = c_{t+1})$,

$$\begin{aligned} \frac{1}{c_t} &= R\left[\beta\delta\lambda + \delta\left(1-\lambda\right)\right]\frac{1}{c_{t+1}} && \text{(Euler Equation)} \\ c_t &= \lambda x = (1-\lambda)R\lambda x = c_{t+1} && \text{(Steady State)} \end{aligned}$$

- Calibrate $\beta = 2/3$, $\delta = 0.99$:

$$\begin{aligned} \lambda &= \frac{1-\delta}{1-\delta(1-\beta)} = 0.015. \\ 1 &= (1-\lambda)R \\ r &= R-1 \simeq \lambda = 0.015. \end{aligned}$$

- Why is the equilibrium interest rate so low?

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