Instantaneous Gratification

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Abstract. We propose a tractable continuous-time model of hyperbolic discounting that can be used to study the behavior of liquidity-constrained consumers. We show that our dynamically *inconsistent* model shares the same value function as a related dynamically *consistent* optimization problem with a wealth contingent utility function. Using this partial equivalence, we can show both existence and *uniqueness* of a hyperbolic equilibrium. We also show that the equilibrium consumption function is continuous and monotonic in wealth. None of these properties apply generally to analogous discrete-time models of hyperbolic discounting. All of the pathological properties of discrete-time hyperbolic models are eliminated by our continuous-time model.

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1. Introduction

Robert Strotz (1956) first suggested that discount rates are higher in the short run than in the long run. Almost every experimental study on time preference has supported his conjecture (Ainslie 1992). To capture this empirical regularity, Laibson (1997a) adopted a discrete-time discount function, $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots.\}$, which Phelps and Pollak (1968) had previously used to model intergenerational time preferences. With $\beta < 1$, this 'hyperbolic' discount function captures the gap between a high short-run discount rate, $-\ln(\beta\delta)$, and a low long-run rate, $-\ln(\delta)$. In the last several years, this discrete-time discount function has been used to model a wide range of behavior: e.g., saving, contracts, job search.[1]

The hyperbolic discount function implies that current preferences are inconsistent with those held in the future. Beginning with the work of Strotz, such dynamic inconsistency has been analyzed by treating the individual as a sequence of independent selves whose choices are modelled as an intrapersonal game.

This game-theoretic framework has proved extremely fruitful. A recurrent problem has, however, plagued most of these hyperbolic applications: strategic interaction among intrapersonal selves often generates counterfactual policy functions. Hyperbolic consumption functions need not be globally monotonic in wealth, and may even drop discontinuously at a countable number of points.

Such non-monotonicities arise because early selves are faced with two competing strategies with very different implications for current consumption. In the first strategy, the early self consumes a lot, thereby depriving later selves of resources that those later selves would wastefully splurge. Alternatively, the early self consumes relatively little, thereby providing later selves with enough resources so that those later selves will be able to both consume *and* save for the future. As wealth rises, the early self switches from the first strategy to the second, generating a negative slope

[1] For examples, see O'Donoghue and Rabin (1999b), Angeletos, Laibson, Repetto, Tobacman and Weinberg (2000), and Della Vigna and Paserman (2000).

in some interval of the consumption function. Often such declines are discontinuous. Numerous authors, including Laibson (1997b), Morris and Postlewaite (1997), O'Donoghue and Rabin (1999a), Harris and Laibson (2000), and Krusell and Smith (2000) have identified hyperbolic examples in which the consumption function has negatively sloped intervals. We call these examples 'hyperbolic pathologies'. Figure 1 plots examples of such pathological consumption functions.

Two classes of solutions have been proposed to address such pathologies. First, Harris and Laibson (2000) point out that such pathologies occur *only* when the model is calibrated in a limited region of the parameter space. When the hyperbolic model is calibrated with reasonable levels of noise (i.e., income volatility) and reasonable values of other preference and technology parameters, the pathologies typically vanish. But Harris and Laibson (2000) acknowledge that there *do* exist defensible calibrations for which the pathologies are still present (notably when the coefficient of relative risk aversion lies well below unity).

Second, O'Donoghue and Rabin (1999a) point out that the pathologies arise *only* when consumers are modelled as rational agents. If consumers do not recognize that their own preferences are dynamically inconsistent, they will not have any incentive to act strategically vis-à-vis their own future selves. Hence, naive consumers who do not anticipate their own dynamic inconsistency will not exhibit pathologies. However, this solution requires that consumers be completely naive about their own future preferences. Any partial knowledge of future dynamic inconsistency reinstates the pathologies.

In the current paper we identify a solution to the pathology problem that is more robust than either of those discussed above. First, we propose a continuous-time model of time discounting that captures the qualitative properties of the hyperbolic model. This model distinguishes between the 'present' and the 'future'. The present is valued discretely more than the future, mirroring the one-time drop in valuation implied by the discrete-time quasi-hyperbolic discount function (Laibson 1997) and its continuous-time generalizations (Barro 1999, Luttmer and Mariotti 2000). In addition, we assume that the transition from the present to the future is determined by a constant hazard rate. This simplifying assumption enables us to reduce our problem to a system of two differential equations that characterize present and future value functions.

We find that this simplified continuous-time model is still not tractable when we allow for a class of general utility functions and liquidity constraints. However, our model has a limit case that is analytically tractable and psychologically relevant. This is the case in which the present is vanishingly short. By focusing on this psychologically important limit case, we take the phrase "instantaneous gratification" literally. We analyze a model in which individuals prefer gratification in the present instant discretely more than consumption in the momentarily delayed future. This model is a useful benchmark that characterizes the neighborhood of models in which the present is short, but not precisely instantaneous.

We show that the instantaneous-gratification model, which is dynamically *inconsistent*, shares the same value function as a related dynamically *consistent* optimization problem with a wealth-contingent utility function. Using this partial equivalence, we can show both existence and *uniqueness* of the hyperbolic equilibrium. We also show that the equilibrium consumption function is continuous and monotonic in wealth. The monotonicity property relies on the condition that the long-run discount rate is weakly greater than the interest rate. *All* of the pathological properties of discrete-time hyperbolic models are eliminated by our continuous-time model.

Two other sets of authors have analyzed hyperbolic preferences in continuous time. Barro (1999) analyzes the choices of hyperbolic agents with constant relative risk aversion preferences, deterministic income paths, and access to perfect markets. He focuses on the general equilibrium implications of hyperbolic discounting and the ways in which hyperbolic economies may be observationally equivalent to exponential economies. Luttmer and Mariotti (2000) begin with the Barro problem and introduce stochastic asset returns. Luttmer-Mariotti extend Barro's observational-equivalence result, but also identify endowment processes for which the hyperbolic model has new asset-pricing implications (e.g., an elevated equity premium). Luttmer and Mariotti work with general discount functions and consider numerous special cases. They have independently identified some special properties of the particular case in which the present is vanishingly short. However, these findings do not overlap with ours.

Barro and Luttmer-Mariotti both focus on linear policy rules, which support a unique equilibrium in their respective models. The existence of this linear equilibrium results from special preference assumptions (constant relative risk aversion), market assumptions (complete markets, including sales of future labor income), and a restriction to linear policy rules. We generalize these restrictive assumptions. We work with a broad class of preferences. We introduce the realistic constraint that consumers may not borrow against future labor income. We do not require linear policy rules. Indeed, our problem does not admit a linear equilibrium. We pursue these generalizations for greater realism. We contend with the resulting pathologies that arise in our general setting but do not arise under the Barro/Luttmer-Mariotti assumptions in either discrete or continuous time.

Our results also differ from Barro and Luttmer-Mariotti in that we are able to prove uniqueness in the class of Markov equilibria. This is a desirable and unexpected result, since the hyperbolic model is a dynamic game, which can generate indeterminacy. For example, Krusell and Smith (2000) have shown that hyperbolic Markov equilibria are *not* unique in a deterministic discrete-time setting. In the current paper, we provide two uniqueness results. First, we prove uniqueness in a class of continuous-time models with stochastic asset returns. Second, we propose a refinement that uses the unique equilibrium in the stochastic setting to select a sensible unique equilibrium in the deterministic setting. This refinement takes the natural approach of selecting the limiting equilibrium obtained as the noise vanishes.

The rest of the paper formalizes our claims. In Section 2 we describe our general continuous-time model. In Section 3 we describe an important limit case of our model. We call this limit case the instantaneous-gratification model. In Section 4 we show that the instantaneous-gratification model has the same *value function* as a particular dynamically consistent optimization problem. We call this latter problem the 'equivalent problem', but note that it is *not* observationally equivalent to the hyperbolic problem. The instantaneous-gratification model shares the same long-run discount rate as the equivalent problem, but the two problems have different instantaneous utility functions and different equilibrium policy functions. In Section 5, we use our partial equivalence result to derive several important properties of the instantaneous-gratification problem, including equilibrium existence, equilibrium uniqueness, consumption-function continuity, and consumption-function monotonicity. In Section 6 we consider the deterministic version of the instantaneous-gratification problem, and derive some additional properties. In Section 7 we reformulate and generalize the results of Section 3 showing that the instantaneous-gratification model is the limit of the general model of Section 2. In Section 8 we conclude.

2. The Model with Finite λ

2.1. Dynamics. At the outset of period $t \in [0, +\infty)$, the consumer has wealth $x \in [0, +\infty)$. She then receives labor income $y \in [0, +\infty)$. If $x > 0$, she may choose any consumption level $c \in (0, +\infty)$. If $x = 0$, she may choose any consumption level $c \in (0, y]$. Whatever she does not consume is invested in an asset, the returns on which are distributed normally with mean μdt and variance $\sigma^2 dt$, where $\mu \in (-\infty, +\infty)$ and $\sigma \in (0, +\infty)$. The change in her wealth in period t is therefore

$$dx = (\mu x + y - c)\, dt + \sigma x dz,$$

where z is a standard Wiener process. In particular, the consumer may never borrow.

2.2. Preferences. The consumer is modelled as a 'sequence' of autonomous selves. These selves are indexed by the period in which they control the consumption choice.

In the standard discrete-time formulation of quasi-hyperbolic preferences the present consists of the single period t. The future consists of periods $t+1$, $t+2$, ... A period n steps into the future is discounted with the factor δ^n and an additional discount factor β is applied to all periods except the present (Laibson 1997). This model can be generalized in two ways. First, the present can last for any number of periods $T_t \in \{1, 2, ...\}$. Secondly, T_t can be random. The preferences in equation (1) below are a natural continuous-time analogue of this more general formulation.

In the present continuous-time setting, we assume that the preferences of self t are given by

$$\mathrm{E}_t \left[\int_t^{t+T_t} e^{-\gamma(s-t)} U\left(c\left(s\right)\right) ds + \alpha \int_{t+T_t}^{+\infty} e^{-\gamma(s-t)} U\left(c\left(s\right)\right) ds \right], \qquad (1)$$

where $\gamma \in (0, +\infty)$, $\alpha \in (0, 1]$, $U : (0, +\infty) \to \mathbb{R}$ and T_t is distributed exponentially with parameter $\lambda \in [0, +\infty)$. In other words, self t uses a stochastic discount function that decays exponentially at rate γ up to time T_t, drops discontinuously at T_t to a fraction α of its level just prior to T_t, and decays exponentially at rate γ thereafter.

$$\text{Discount function } \equiv D(t, s) = \begin{cases} e^{-\gamma(s-t)} & \text{if} \quad s \leq T_t \\ \alpha e^{-\gamma(s-t)} & \text{if} \quad s > T_t \end{cases}$$

This continuous-time formalization is close to the deterministic functions used in Barro (1999) and Luttmer and Mariotti (2000). However, we assume T_t is stochastic. The stochastic transition with constant hazard rate enables us to reduce our problem to a system of two differential equations that characterize present and future value functions.

Figure 2 plots a single realization of this discount function, with $T_t = 3.4$. Figure

3 plots the expected value of the discount function for a range of λ values: $\lambda \in \{0, 0.1, 1, 10, \infty\}$. Analytically, the expected value is given by,

$$\mathrm{E}_t D(t,s) = e^{-\lambda(s-t)} e^{-\gamma(s-t)} + (1 - e^{-\lambda(s-t)}) \alpha e^{-\gamma(s-t)}.$$

When $\lambda = 0$ our discount function is equivalent to a standard exponential discount function. As $\lambda \to \infty$ our discount function converges to a jump function:

$$\lim_{\lambda\to\infty} D(t,s) = \begin{cases} 1 & \text{if} \quad s = t \\ \alpha e^{-\gamma(s-t)} & \text{if} \quad s > t \end{cases}$$

Letting λ go to infinity captures the special case in which the present is vanishingly short. We will return to this case below.

2.3. Equilibrium. We confine attention to the set of perfect equilibria in stationary Markov strategies. Moreover we focus on equilibria that are bounded below in the sense that the continuation-value function is at least $\frac{U(y)}{\gamma}$.

2.4. Discussion of the Model. Our continuous-time buffer-stock model has an immediate advantage over its discrete-time analogue: any equilibrium consumption function C is continuous everywhere on $[0, +\infty)$. However, the principal pathology of the discrete-time buffer-stock model remains: while C is perfectly well behaved for small values of x, the average propensity to consume fluctuates for large values of x. These fluctuations may be large enough to produce intervals in which $C' < 0$.

This pathology can be particularly well illustrated in the limiting case of the model in which $\sigma = 0$. In this case, the Bellman system is essentially two-dimensional. It is therefore possible to draw a phase portrait for it. This portrait shows that, as $x \to +\infty$, the system spirals around a steady state. This steady state is simply the solution of the homogeneous model obtained by putting $y = 0$. In particular, the average propensity to consume oscillates around its steady-state value. This automatically means that C cannot be concave; and, in extreme cases, it means that C may not even be increasing.

Fortunately, we need not be interested in the general case of this model. The urge for "instantaneous gratification" suggests that the present — i.e., the interval from t to $t + T_t$ during which consumption is particularly highly valued — is very short. These observations lead us to consider the limiting case of the model in which $\lambda \to +\infty$, and hence the present becomes vanishingly short. We refer to this limiting case as the instantaneous-gratification model.

3. The Instantaneous-Gratification Model

3.1. Working Hypotheses. Let $W : [0, +\infty) \to \mathbb{R}$ be the current-value function, let $V : [0, +\infty) \to \mathbb{R}$ be the continuation-value function and let $C : [0, +\infty) \to [0, +\infty)$ be the consumption function. Then we make the following working hypotheses:

H1 W and V are continuous on $[0, +\infty)$;

H2 W and V are twice continuously differentiable in $(0, +\infty)$;

H3 $W'(0+)$, $W''(0+)$, $V'(0+)$ and $V''(0+)$ all exist;

H4 there exist continuous $\overline{W}, \overline{V} : [0, +\infty) \to \mathbb{R}$ such that $W \to \overline{W}$ and $V \to \overline{V}$ uniformly on compact subsets of $[0, +\infty)$ as $\lambda \to +\infty$;

H5 $\overline{W}$ and $\overline{V}$ are twice continuously differentiable in $(0, +\infty)$;

H6 $\overline{W}'(0+)$, $\overline{W}''(0+)$, $\overline{V}'(0+)$ and $\overline{V}''(0+)$ all exist;

H7 $W' \to \overline{W}'$, $W'' \to \overline{W}''$, $V' \to \overline{V}'$ and $V'' \to \overline{V}''$ uniformly on compact subsets of $(0, +\infty)$ as $\lambda \to +\infty$;

H8 there exists $K \in [0, +\infty)$ such that, for all $\lambda \in [0, +\infty)$, $C(x) \leq K(1 + x)$.

Moreover we put $W'(0) = W'(0+)$, $W''(0) = W''(0+)$, $V'(0) = V'(0+)$, $V''(0) = V''(0+)$, $\overline{W}'(0) = \overline{W}'(0+)$, $\overline{W}''(0) = \overline{W}''(0+)$, $\overline{V}'(0) = \overline{V}'(0+)$ and $\overline{V}''(0) = \overline{V}''(0+)$.

Remark 1. *Note that while W and V are assumed to converge uniformly on compact subsets of $[0, +\infty)$, W', W'', V' and V'' are only assumed to converge uniformly on compact subsets of $(0, +\infty)$. This distinction is crucial: it turns out that, while C is continuous on the whole of $[0, +\infty)$, its limit $\overline{C}$ may have an upward jump at 0.*

3.2. The Bellman System of the Model with Finite λ. For all $\phi \in (0, +\infty)$, put

$$f(\phi) = \operatorname*{argmax}_{c \in (0,+\infty)} U(c) - c\phi \text{ and } f_0(\phi) = \operatorname*{argmax}_{c \in (0,y]} U(c) - c\phi.$$

Then:

$$\frac{1}{2}\sigma^2 x^2 W'' + (\mu x + y - C) W' - \gamma W - \lambda (W - \alpha V) + U(C) = 0, \tag{2}$$

$$\frac{1}{2}\sigma^2 x^2 V'' + (\mu x + y - C) V' - \gamma V + U(C) = 0 \tag{3}$$

and

$$C = f(W') \tag{4}$$

when $x > 0$; and

$$(y - C) W' - \gamma W - \lambda (W - \alpha V) + U(C) = 0, \tag{5}$$

$$(y - C) V' - \gamma V + U(C) = 0 \tag{6}$$

and

$$C = f_0(W') \tag{7}$$

when $x = 0$. We refer to this system as the Bellman system with finite λ.

3.3. Continuity of the Consumption Function. It follows at once from the continuity of W' that C is continuous in $(0,+\infty)$. For all $\phi \in (0,+\infty)$, put

$$h(\phi) = \max_{c \in (0,+\infty)} U(c) - c\phi \text{ and } h_0(\phi) = \max_{c \in (0,y]} U(c) - c\phi.$$

Then equation (2) can be written

$$\frac{1}{2}\sigma^2 x^2 W'' + (\mu x + y) W' - \gamma W - \lambda (W - \alpha V) + h(W') = 0, \tag{8}$$

and equation (5) can be written

$$yW' - \gamma W - \lambda (W - \alpha V) + h_0(W') = 0. \tag{9}$$

Letting $x \to 0+$ in equation (8), we obtain

$$yW' - \gamma W - \lambda (W - \alpha V) + h(W') = 0 \tag{10}$$

when $x = 0$. Now $h(W') \geq h_0(W')$ for all W', with strict inequality if $W' < U'(y)$. Comparing equation (9) with equation (10), we therefore have $W' \geq U'(y)$. Hence C is continuous at 0 as well.

3.4. Derivation of the Bellman System of the Instantaneous-Gratification Model. Put $Z = W - \alpha V$. Multiplying equations (3) and (6) by α and subtracting them from equations (2) and (5), we obtain

$$\frac{1}{2}\sigma^2 x^2 Z'' + (\mu x + y - C) Z' - (\gamma + \lambda) Z + (1 - \alpha) U(C) = 0$$

when $x \geq 0$. In other words, Z is the e.p.d.v. of the flow of utility $(1-\alpha) U(C)$ up to time T_t. Hence $Z \to 0$ as $\lambda \to +\infty$, and $\overline{W} = \alpha \overline{V}$.

Next, put $\overline{C} = f\left(\alpha \overline{V}'\right)$ for all $x > 0$. Then, passing to the limit in equation (3), we obtain

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + \left(\mu x + y - \overline{C}\right)\overline{V}' - \gamma\overline{V} + U\left(\overline{C}\right) = 0 \tag{11}$$

when $x > 0$.

Next, let x^0 be the time-path for assets starting at 0. Then we have:

$$dx^0(t) = \left(\mu x^0(t) + y - C\left(x^0(t)\right)\right)dt + \sigma x^0(t)\,dz(t),$$

with initial condition

$$x^0(0) = 0;$$

and

$$V(0) = \mathrm{E}\left[\int e^{-\gamma t} U\left(C\left(x^0(t)\right)\right)dt\right].$$

Let $\overline{x}^0$ be any limit point of x^0. Then there are two cases to consider.

Suppose first that $\overline{V}'(0) \leq \frac{U'(y)}{\alpha}$. In this case $\overline{C}(0+) \geq y$, and $\overline{x}^0$ must remain trapped at 0 forever. We can therefore find $\kappa : [0, +\infty) \to \mathcal{P}([0, K])$ such that

$$d\overline{x}^0(t) = y - \int c\,d\kappa(c \mid t) = 0$$

for all $t \geq 0$ and

$$\overline{V}(0) = \mathrm{E}\left[\int e^{-\gamma t}\left(\int U(c)\,d\kappa(c \mid t)\right)dt\right].$$

Hence

$$\frac{U(y)}{\gamma} \leq \overline{V}(0) \leq \mathrm{E}\left[\int e^{-\gamma t} U\left(\int c\,d\kappa(c \mid t)\right)dt\right] = \mathrm{E}\left[\int e^{-\gamma t} U(y)\,dt\right] = \frac{U(y)}{\gamma}.$$

Hence: $\overline{V}(0) = \frac{U(y)}{\gamma}$; and $\kappa(\cdot \mid t)$ assigns probability 1 to y for almost all $t \geq 0$. In particular, $\overline{C}(0)$ is well defined and equal to y.

Suppose second that $\overline{V}'(0) > \frac{U'(y)}{\alpha}$. In this case $\overline{C}(0+) < y$, and once $\overline{x}^0$ enters the open interval $(0, +\infty)$, it must remain there forever. We can therefore find a stopping time τ and a $\kappa : [0, \tau) \to \mathcal{P}([0, K])$ such that

$$d\overline{x}^0(t) = \left\{ \begin{array}{ll} y - \int c d\kappa(c \mid t) = 0 & \text{for all } t \in [0, \tau) \\ \left(\mu \overline{x}^0(t) + y - \overline{C}\left(\overline{x}^0(t)\right)\right) dt + \sigma \overline{x}^0(t)\, dz(t) & \text{for all } t \in (\tau, +\infty) \end{array} \right\}$$

and

$$\overline{V}(0) = \mathrm{E}\left[\int_0^\tau e^{-\gamma t} \left(\int U(c)\, d\kappa(c \mid t)\right) dt + \int_\tau^{+\infty} e^{-\gamma t} U\left(\overline{C}\left(\overline{x}^0(t)\right)\right) dt\right].$$

Hence

$$\begin{aligned} \overline{V}(0) &\le \mathrm{E}\left[\int_0^\tau e^{-\gamma t} U\left(\int c d\kappa(c \mid t)\right) dt + \int_\tau^{+\infty} e^{-\gamma t} U\left(\overline{C}\left(\overline{x}^0(t)\right)\right) dt\right] \\ &= \mathrm{E}\left[\int_0^\tau e^{-\gamma t} U(y)\, dt + e^{-\gamma \tau} \overline{V}(0)\right] = \mathrm{E}\left[\left(1 - e^{-\gamma \tau}\right) \frac{U(y)}{\gamma} + e^{-\gamma \tau} \overline{V}(0)\right] \\ &= \frac{U(y)}{\gamma} + \mathrm{E}\left[e^{-\gamma \tau}\right] \left(\overline{V}(0) - \frac{U(y)}{\gamma}\right) \end{aligned}$$

and

$$\left(\overline{V}(0) - \frac{U(y)}{\gamma}\right) \le \mathrm{E}\left[e^{-\gamma \tau}\right] \left(\overline{V}(0) - \frac{U(y)}{\gamma}\right). \tag{12}$$

Now, letting $x \to 0+$ in equation (11), we obtain

$$\left(y - \overline{C}(0+)\right) \overline{V}'(0) - \gamma \overline{V}(0) + U\left(\overline{C}(0+)\right) = 0$$

and

$$\begin{aligned} \overline{V}(0) &= \frac{1}{\gamma}\left(y \overline{V}'(0) + U\left(\overline{C}(0+)\right) - \overline{C}(0+) \overline{V}'(0)\right) \\ &> \frac{1}{\gamma}\left(y \overline{V}'(0) + U(y) - y \overline{V}'(0)\right) = \frac{U(y)}{\gamma}. \end{aligned} \tag{13}$$

Hence, combining equations (12) and (13), we obtain

$$\mathrm{E}\left[e^{-\gamma\tau}\right] \geq 1.$$

In other words, $\tau = 0$ with probability 1. In particular, we effectively have $\overline{C}(0) = \overline{C}(0+)$.

3.5. The Bellman System of the Instantaneous-Gratification Model. To summarize, when $\lambda \to +\infty$, the Bellman system with finite λ converges to the system:

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + \left(\mu x + y - \overline{C}\right)\overline{V}' - \gamma\overline{V} + U\left(\overline{C}\right) = 0 \tag{14}$$

and

$$\overline{C} = f\left(\alpha\overline{V}'\right) \tag{15}$$

when $x > 0$; and

$$\left(y - \overline{C}\right)\overline{V}' - \gamma\overline{V} + U\left(\overline{C}\right) = 0 \tag{16}$$

and

$$\overline{C} = f_0\left(\alpha\overline{V}'\right) \tag{17}$$

when $x = 0$. We refer to this new system as the Bellman system of the instantaneous-gratification model.

Remark 2. *Our derivation of the boundary condition for the Bellman system of the instantaneous-gratification model highlights the importance of the requirement that* $V \geq \frac{U(y)}{\gamma}$.

4. An Equivalence Result

In the present section we show that, under appropriate assumptions, the value function $\overline{V}$ of the hyperbolic consumer in the instantaneous-gratification model is also the value function of an exponential consumer with appropriately chosen utility function.

This result can be motivated by comparing the Bellman system with $\lambda = 0$ with the Bellman system of the instantaneous-gratification model. Recall first that, for all $\phi \in (0, +\infty)$, we have

$$h(\phi) = \max_{c \in (0,+\infty)} U(c) - c\phi \text{ and } h_0(\phi) = \max_{c \in (0,y]} U(c) - c\phi.$$

Hence, putting $\lambda = 0$ in the Bellman system with finite λ, we obtain:

$$\frac{1}{2}\sigma^2 x^2 W'' + (\mu x + y) W' - \gamma W + h(W') = 0 \tag{18}$$

when $x > 0$; and

$$yW' - \gamma W + h_0(W') = 0 \tag{19}$$

when $x = 0$. Secondly, for all $\phi \in (0, +\infty)$, put

$$\widehat{h}(\phi) = U(f(\alpha\phi)) - f(\alpha\phi)\phi \text{ and } \widehat{h}_0(\phi) = U(f_0(\alpha\phi)) - f_0(\alpha\phi)\phi.$$

Then the Bellman system of the instantaneous-gratification model can be written:

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + (\mu x + y)\overline{V}' - \gamma\overline{V} + \widehat{h}\left(\overline{V}'\right) = 0 \tag{20}$$

when $x > 0$; and

$$y\overline{V}' - \gamma\overline{V} + \widehat{h}_0\left(\overline{V}'\right) = 0 \tag{21}$$

when $x = 0$.

Now, provided that $\widehat{h}$ is decreasing and convex, we can find a utility function $\widehat{U}$ such that

$$\widehat{h}(\phi) = \max_{c \in (0,+\infty)} \widehat{U}(c) - c\phi.$$

Similarly, provided that $\widehat{h}_0$ is decreasing and convex, we can find a utility function

$\widehat{U}_0$ such that

$$\widehat{h}_0(\phi) = \max_{c \in (0,y]} \widehat{U}_0(c) - c\phi.$$

However, unlike h and h_0, $\widehat{h}$ and $\widehat{h}_0$ are not generated by the same utility function. On the contrary, $\widehat{U}_0$ dominates $\widehat{U}$. In particular, we have $\widehat{U}_0(y) > \widehat{U}(y)$. Hence, in order to obtain the desired equivalence, the utility function of the exponential consumer must be made to depend on her wealth as her consumption. Specifically, she must use the utility function $\widehat{U}$ when $x > 0$ and the utility function $\widehat{U}_0$ when $x = 0$.

4.1. Assumptions. In order to establish our equivalence result, we shall need the following basic assumptions:

A1 U has domain $(0, +\infty)$ and range $(-\infty, +\infty)$;

A2 U is twice continuously differentiable on $(0, +\infty)$;

A3 $U'(c) > 0$ for all $c \in (0, +\infty)$;

A4 there exist $0 < \underline{\rho}_U \leq \overline{\rho}_U < +\infty$ such that $\underline{\rho}_U \leq \frac{-cU''(c)}{U'(c)} \leq \overline{\rho}_U$ for all $c \in (0, +\infty)$;

A5 $\gamma > \max\left\{\left(1 - \underline{\rho}_U\right)\left(\mu - \frac{1}{2}\underline{\rho}_U \sigma^2\right), \left(1 - \overline{\rho}_U\right)\left(\mu - \frac{1}{2}\overline{\rho}_U \sigma^2\right)\right\}$.

Assumptions A1-A4 can be summarized by saying that the consumer has bounded relative risk aversion, or BRRA for short. Assumption A5 is the natural integrability condition for an exponential consumer with BRRA preferences: it ensures that the expected payoff of such a consumer is well defined.

We shall also need the following further assumptions, which are specific to the hyperbolic context:

B1 there exist $-\infty < \underline{\theta}_U \leq \overline{\theta}_U < +\infty$ such that $\underline{\theta}_U \leq \frac{U'(c)U'''(c)}{U''(c)^2} \leq \overline{\theta}_U$ for all $c \in (0, +\infty)$;

B2 $\alpha + \underline{\rho}_U - 1 > 0$;

B3 $(2-\alpha)-(1-\alpha)\,\overline{\theta}_U>0.$

Assumption B1 requires that the ratio of the coefficient of relative prudence $\frac{-cU'''(c)}{U''(c)}$ to the coefficient of relative risk aversion $\frac{-cU''(c)}{U'(c)}$ is bounded away from 0 and $+\infty$; Assumption B2 ensures that $\widehat{h}$ and $\widehat{h}_0$ are decreasing; and Assumption B3 ensures that $\widehat{h}$ and $\widehat{h}_0$ are convex.

Remark 3. *Assumptions A1-A5 and B1-B3 can be dramatically simplified if U has a constant coefficient of relative risk aversion ρ_U: Assumptions A1-A5 reduce to the requirement that $\gamma>(1-\rho_U)\left(\mu-\frac{1}{2}\rho_U\sigma^2\right)$; and Assumptions B1-B3 reduce to the requirement that $\alpha+\rho_U-1>0$.*

4.2. Analysis of $\widehat{h}$. Recall that $\widehat{h}(\phi)=U(f(\alpha\phi))-f(\alpha\phi)\,\phi$ for all $\phi\in(0,+\infty)$.

Theorem 4. *Suppose that Assumptions A1-A5 and B1-B3 hold. Then:*

1. $\widehat{h}'(\phi)<0$ *for all* $\phi\in(0,+\infty)$;
2. $\widehat{h}''(\phi)>0$ *for all* $\phi\in(0,+\infty)$;
3. *there exist* $0<\underline{\rho}_{\widehat{h}}\leq\overline{\rho}_{\widehat{h}}<+\infty$ *such that* $\underline{\rho}_{\widehat{h}}\leq\frac{-\phi\widehat{h}''(\phi)}{\widehat{h}'(\phi)}\leq\overline{\rho}_{\widehat{h}}$ *for all* $\phi\in(0,+\infty)$.

Proof. Note first that

$$\widehat{h}(\phi)=U(f(\alpha\phi))-f(\alpha\phi)\,\phi=U(f(\alpha\phi))-\alpha f(\alpha\phi)\,\phi-(1-\alpha)\,f(\alpha\phi)\,\phi$$

$$=h(\alpha\phi)-(1-\alpha)\,f(\alpha\phi)\,\phi.$$

Hence

$$\widehat{h}'(\phi)=\alpha h'(\alpha\phi)-(1-\alpha)\,f(\alpha\phi)-(1-\alpha)\,\alpha f'(\alpha\phi)\,\phi$$

$$=-\alpha f(\alpha\phi)-(1-\alpha)\,f(\alpha\phi)-(1-\alpha)\,\alpha f'(\alpha\phi)\,\phi=-f(\alpha\phi)-(1-\alpha)\,f'(\alpha\phi)\,\alpha\phi$$

$$= -f(\alpha\phi)\left(1 + (1-\alpha)\frac{f'(\alpha\phi)\,\alpha\phi}{f(\alpha\phi)}\right) = -f(\alpha\phi)\left(1 + (1-\alpha)\frac{U'(f(\alpha\phi))}{f(\alpha\phi)\,U''(f(\alpha\phi))}\right)$$

$$= -f(\alpha\phi)\left(1 - \frac{1-\alpha}{\rho_U(f(\alpha\phi))}\right) = \frac{-f(\alpha\phi)}{\rho_U(f(\alpha\phi))}\left(\alpha + \rho_U(f(\alpha\phi)) - 1\right).$$

Part 1 of the Theorem therefore follows from Assumption B2.

Secondly, as shown above, we have

$$\widehat{h}'(\phi) = -f(\alpha\phi) - (1-\alpha)\,f'(\alpha\phi)\,\alpha\phi.$$

Hence

$$\begin{aligned}\widehat{h}''(\phi) &= -\alpha f'(\alpha\phi) - (1-\alpha)\,f'(\alpha\phi)\,\alpha - (1-\alpha)\,\alpha f''(\alpha\phi)\,\alpha\phi \\ &= -\alpha f'(\alpha\phi)\left(1 + (1-\alpha)\left(1 + \frac{f''(\alpha\phi)\,\alpha\phi}{f'(\alpha\phi)}\right)\right) \\ &= \frac{-\alpha}{U''(f(\alpha\phi))}\left(1 + (1-\alpha)\left(1 - \frac{U'''(f(\alpha\phi))\,U'(f(\alpha\phi))}{U''(f(\alpha\phi))^2}\right)\right) \\ &= \frac{-\alpha}{U''(f(\alpha\phi))}\left(1 + (1-\alpha)\left(1 - \theta_U(f(\alpha\phi))\right)\right) \\ &= \frac{-\alpha}{U''(f(\alpha\phi))}\left((2-\alpha) - (1-\alpha)\,\theta_U(f(\alpha\phi))\right).\end{aligned}$$

Part 2 of the Theorem therefore follows from Assumption B3.

Thirdly, using the final expressions obtained above for $\widehat{h}'(\phi)$ and $\widehat{h}''(\phi)$, we have

$$\frac{-\phi\widehat{h}''(\phi)}{\widehat{h}'(\phi)} = \frac{(2-\alpha) - (1-\alpha)\,\theta_U(f(\alpha\phi))}{(\alpha + \rho_U(f(\alpha\phi)) - 1)}.$$

Hence

$$\frac{(2-\alpha) - (1-\alpha)\,\overline{\theta}_U}{(\alpha + \overline{\rho}_U - 1)} \leq \frac{-\phi\widehat{h}''(\phi)}{\widehat{h}'(\phi)} \leq \frac{(2-\alpha) - (1-\alpha)\,\underline{\theta}_U}{\left(\alpha + \underline{\rho}_U - 1\right)}.$$

We may therefore put

$$\underline{\rho}_{\widehat{h}} = \frac{(2-\alpha) - (1-\alpha)\,\overline{\theta}_U}{(\alpha + \overline{\rho}_U - 1)} \text{ and } \overline{\rho}_{\widehat{h}} = \frac{(2-\alpha) - (1-\alpha)\,\underline{\theta}_U}{\left(\alpha + \underline{\rho}_U - 1\right)}.$$

This establishes part 3 of the Theorem. ■

We may therefore apply Fenchel's Theorem to conclude that, if we define the function $\widehat{U} : (0, +\infty) \to \mathbb{R}$ by the formula

$$\widehat{U}\,(\widehat{c}) = \min_{\phi \in (0,+\infty)} \widehat{h}\,(\phi) + \widehat{c}\phi,$$

then

$$\widehat{h}\,(\phi) = \max_{\widehat{c} \in (0,+\infty)} \widehat{U}\,(\widehat{c}) - \phi\widehat{c}.$$

Moreover:

Theorem 5. *Suppose that Assumptions A1-A5 and B1-B3 hold. Then:*

1. $\widehat{U}'\,(\widehat{c}) > 0$ *for all* $\widehat{c} \in (0, +\infty)$;
2. $\widehat{U}''\,(\widehat{c}) < 0$ *for all* $\widehat{c} \in (0, +\infty)$;
3. *there exist* $0 < \underline{\rho}_{\widehat{U}} \leq \overline{\rho}_{\widehat{U}} < +\infty$ *such that* $\underline{\rho}_{\widehat{U}} \leq \frac{-\widehat{c}\widehat{U}''(\widehat{c})}{\widehat{U}'(\widehat{c})} \leq \overline{\rho}_{\widehat{U}}$ *for all* $\phi \in (0, +\infty)$.

Proof. Put

$$\widehat{g}\,(\widehat{c}) = \operatorname{argmin}_{\phi \in (0,+\infty)} \widehat{h}\,(\phi) + \widehat{c}\phi.$$

Then

$$\widehat{U}'\,(\widehat{c}) = \widehat{g}\,(\widehat{c}),$$

$$\widehat{U}''\,(\widehat{c}) = \widehat{g}'\,(\widehat{c}) = \frac{-1}{\widehat{h}''\,(\widehat{g}\,(\widehat{c}))}$$

and

$$\frac{-\widehat{c}\widehat{U}''\,(\widehat{c})}{\widehat{U}'\,(\widehat{c})} = \frac{-\widehat{h}'\,(\widehat{g}\,(\widehat{c}))}{\widehat{g}\,(\widehat{c})\,\widehat{h}''\,(\widehat{g}\,(\widehat{c}))}.$$

In particular, we may put

$$\underline{\rho}_{\widehat{U}} = \frac{1}{\overline{\rho}_{\widehat{h}}} \text{ and } \overline{\rho}_{\widehat{U}} = \frac{1}{\underline{\rho}_{\widehat{h}}}.$$

This completes the proof of the Theorem. ■

4.3. Analysis of $\widehat{h}_0$. Recall that $\widehat{h}_0(\phi) = U(f_0(\alpha\phi)) - f_0(\alpha\phi)\phi$ for all $\phi \in (0, +\infty)$.

Theorem 6. *Suppose that Assumptions A1-A5 and B1-B3 hold. Then*

$$\widehat{h}_0(\phi) = \left\{ \begin{array}{cc} U(y) - \phi y & \text{if } 0 < \phi \leq \frac{U'(y)}{\alpha} \\ \widehat{h}(\phi) & \text{if } \frac{U'(y)}{\alpha} < \phi < +\infty \end{array} \right\}.$$

Moreover $\widehat{h}_0'\left(\frac{U'(y)}{\alpha}-\right) \leq \widehat{h}_0'\left(\frac{U'(y)}{\alpha}+\right)$. *In particular,* $\widehat{h}_0$ *is strictly decreasing and convex.*

Proof. The first statement is immediate from the definition of $\widehat{h}_0$. It implies that

$$\widehat{h}_0'\left(\frac{U'(y)}{\alpha}+\right) = \widehat{h}'\left(\frac{U'(y)}{\alpha}\right) = \frac{-f(U'(y))}{\rho_U(f(U'(y)))}(\alpha + \rho_U(f(U'(y))) - 1)$$

$$= -y\left(\frac{\alpha + \rho_U(y) - 1}{\rho_U(y)}\right) \geq -y = \widehat{h}_0'\left(\frac{U'(y)}{\alpha}-\right). ■$$

We may therefore apply Fenchel's Theorem to conclude that, if we define the function $\widehat{U}_0 : (0, y] \to \mathbb{R}$ by the formula

$$\widehat{U}_0(\widehat{c}) = \min_{\phi \in (0,+\infty)} \widehat{h}_0(\phi) + \widehat{c}\phi,$$

then

$$\widehat{h}_0(\phi) = \max_{\widehat{c} \in (0,y]} \widehat{U}_0(\widehat{c}) - \phi\widehat{c}.$$

Moreover:

Theorem 7. *Suppose that Assumptions A1-A5 and B1-B3 hold. Then*

$$\widehat{U}_0\left(\widehat{c}\right)=\left\{\begin{array}{cc}\widehat{U}\left(\widehat{c}\right) & \text{if } 0<\widehat{c}<\psi y \\ \widehat{U}\left(\psi y\right)+\left(\widehat{c}-\psi y\right)\widehat{U}'\left(\psi y\right) & \text{if } \psi y\leq\widehat{c}\leq y\end{array}\right\},$$

where

$$\psi=\frac{\alpha+\rho_U\left(y\right)-1}{\rho_U\left(y\right)}.$$

Moreover $\widehat{U}_0\left(y\right)=U\left(y\right)$.

Proof. We have

$$\widehat{h}_0\left(\phi\right)=\left\{\begin{array}{ll}U\left(y\right)-\phi y & \text{if } 0<\phi\leq\frac{U'(y)}{\alpha} \\ \widehat{h}\left(\phi\right) & \text{if } \frac{U'(y)}{\alpha}<\phi<+\infty\end{array}\right\}.$$

and

$$\widehat{h}_0'\left(\phi\right)\left\{\begin{array}{ll}=-y & \text{if } 0<\phi<\frac{U'(y)}{\alpha} \\ \in\left[-y,-\psi y\right] & \text{if } \phi=\frac{U'(y)}{\alpha} \\ =\widehat{h}'\left(\phi\right) & \text{if } \frac{U'(y)}{\alpha}<\phi<+\infty\end{array}\right\}.$$

Hence

$$\min_{\phi\in(0,+\infty)}\widehat{h}_0\left(\phi\right)+\widehat{c}\phi=\left\{\begin{array}{ll}\min_{\phi\in(0,+\infty)}\widehat{h}\left(\phi\right)+\widehat{c}\phi & \text{if } 0<\widehat{c}<\psi y \\ \widehat{h}_0\left(\frac{U'(y)}{\alpha}\right)+\widehat{c}\frac{U'(y)}{\alpha} & \text{if } \psi y\leq\widehat{c}\leq y \\ -\infty & \text{if } y<\widehat{c}<+\infty\end{array}\right\}.$$

Moreover

$$\min_{\phi\in(0,+\infty)}\widehat{h}\left(\phi\right)+\widehat{c}\phi=\widehat{U}\left(\widehat{c}\right)$$

and

$$\widehat{h}_0\left(\frac{U'\left(y\right)}{\alpha}\right)+\widehat{c}\frac{U'\left(y\right)}{\alpha}=\widehat{h}\left(\frac{U'\left(y\right)}{\alpha}\right)+\widehat{c}\frac{U'\left(y\right)}{\alpha}$$

$$= \widehat{h}\left(\frac{U'(y)}{\alpha}\right) + \psi y \frac{U'(y)}{\alpha} + (\widehat{c} - \psi y)\frac{U'(y)}{\alpha} = \widehat{U}(\psi y) + (\widehat{c} - \psi y)\widehat{U}'(\psi y).$$

Finally,

$$\widehat{h}_0\left(\frac{U'(y)}{\alpha}\right) + y\frac{U'(y)}{\alpha} = U(y) - y\frac{U'(y)}{\alpha} + y\frac{U'(y)}{\alpha} = U(y). \blacksquare$$

4.4. The Equivalent Consumption Problem. The analysis of Sections 4.2 and 4.3 shows that $\overline{V}$ is the value function for the consumption problem of a consumer whose wealth evolves according to the same dynamics as in the original problem, but whose preferences are given by

$$\mathrm{E}_t\left[\int_t^{+\infty} e^{-\gamma(s-t)}\left(\chi_{\{\widehat{x}(s)=0\}}\widehat{U}_0(\widehat{c}(s)) + \chi_{\{\widehat{x}(s)>0\}}\widehat{U}(\widehat{c}(s))\right)ds\right].$$

In other words, the equivalent consumer uses a standard discount function that decays exponentially at rate γ, but uses a non-standard utility function that depends on her wealth.

Remark 8. *We denote consumption and wealth in the equivalent problem by* $\widehat{c}$ *and* $\widehat{x}$ *in order to emphasize the fact that the equivalent consumer makes different consumption choices from the original hyperbolic consumer.*

Figure 4 plots an example of $\widehat{U}_0$ and $\widehat{U}$ for the special case in which U has a constant coefficient of relative risk aversion ρ_U. For this special case we have closed-form solutions,

$$\widehat{U}(\widehat{c}) = \frac{\psi^{\rho_U} U(c)}{\alpha},$$

$$\widehat{U}_0(\widehat{c}) = \left\{\begin{array}{ll} \widehat{U}(\widehat{c}) & \text{if } 0 < \widehat{c} < \psi y \\ \widehat{U}(\psi y) + (\widehat{c} - \psi y)\alpha^{-1} & \text{if } \psi y \leq \widehat{c} \leq y \end{array}\right\},$$

and

$$\psi = \frac{\alpha + \rho_U - 1}{\rho_U}.$$

5. Some Features of the Instantaneous-Gratification Model

In the present section, we exploit the equivalence result of Section 4 to investigate the instantaneous-gratification model. We establish the existence and uniqueness of equilibrium, the continuity of the consumption function in the interior of the wealth space, a sufficient condition for the monotonicity of the consumption function, a generalized Euler equation governing the evolution of the marginal utility of consumption and a corresponding equation governing the evolution of consumption itself. Assumptions A1-A5 and B1-B3 will be in force throughout the section.

5.1. Existence and Uniqueness of Equilibrium.

Theorem 9. *The Bellman system of the instantaneous-gratification model has a unique solution* $\left(\overline{V}, \overline{C}\right)$.

Proof. The equivalence result of Section 4 shows that $\left(\overline{V}, \overline{C}\right)$ solves the Bellman system of the instantaneous-gratification model iff $\overline{V}$ solves the Bellman equation of the equivalent problem, $\overline{C} = f\left(\alpha \overline{V}'\right)$ when $x > 0$ and $\overline{C} = f_0\left(\alpha \overline{V}'\right)$ when $x = 0$. Moreover standard considerations show that the Bellman equation of the equivalent problem possesses a unique solution. ■

5.2. Continuity of the Consumption Function.

Theorem 10. *We have:*

1. $\overline{C}$ *is continuous when* $x > 0$*;*
2. *there exists* $\mu_{crit} \in (-\infty, +\infty)$ *such that* $\overline{C}(0) < \overline{C}(0+)$ *for all* $\mu < \mu_{crit}$ *and* $\overline{C}(0) = \overline{C}(0+)$ *for all* $\mu \geq \mu_{crit}$.

Proof. Note first that $\overline{C}$ is continuous in the interior because $\overline{C} = f\left(\overline{V}'\right)$ there. Secondly, $\overline{C}(0) = f_0\left(\alpha \overline{V}'(0)\right) = y \wedge f\left(\alpha \overline{V}'(0)\right) \leq f\left(\alpha \overline{V}'(0)\right) = \overline{C}(0+)$. Hence $\overline{C}(0) \leq \overline{C}(0+)$, with equality iff $\overline{V}'(0) \geq \frac{U'(y)}{\alpha}$. Thirdly, let $\widetilde{V}$ be the value function of the restricted version of the equivalent consumption problem in which the consumer has utility function $\widehat{U}$ instead of $\widehat{U}_0$ when her wealth is 0. It can be shown that $\overline{V}'(0) \geq \frac{U'(y)}{\alpha}$ iff $\widetilde{V}(0) \geq \frac{U(y)}{\gamma}$. Moreover: $\widetilde{V}(0)$ is strictly increasing in μ; $\widetilde{V}(0) = \frac{\widehat{U}(y)}{\gamma} < \frac{U(y)}{\gamma}$ for all μ sufficiently small; and $\widetilde{V}(0) \to +\infty$ as $\mu \to +\infty$. ■

5.3. Monotonicity of the Consumption Function.

Theorem 11. *Suppose that $\gamma \geq \mu$. Then $\overline{C}' > 0$ when $x > 0$.*

Proof. Note first that $\overline{C} = f\left(\overline{V}'\right)$ in the interior. Hence $\overline{C}$ is continuously differentiable there, and $\overline{C}' = f'\left(\overline{V}'\right)\overline{V}''$. Hence $\overline{C}' > 0$ iff $\overline{V}'' < 0$. Secondly, differentiating equation (20) with respect to x, we obtain

$$\frac{1}{2}\sigma^2 x^2 \overline{V}''' + (\mu x + y)\overline{V}'' - \gamma\overline{V}' + \sigma^2 x\overline{V}'' + \mu\overline{V}' + \widehat{h}'\left(\overline{V}'\right)\overline{V}'' = 0$$

or

$$\overline{V}''' = \frac{2}{\sigma^2 x^2}\left((\gamma - \mu)\overline{V}' - \left(\left(\mu + \sigma^2\right)x + y + \widehat{h}'\left(\overline{V}'\right)\right)\overline{V}''\right).$$

In particular, if $\overline{V}'' = 0$, then

$$\overline{V}''' = \frac{2}{\sigma^2 x^2}(\gamma - \mu)\overline{V}' \geq 0.$$

Hence, if there exists $x_1 \in (0, +\infty)$ such that $\overline{V}''(x_1) \geq 0$, then $\overline{V}'' \geq 0$ on $(x_1, +\infty)$. Thirdly, if there exists $x_1 \in (0, +\infty)$ such that $\overline{V}''(x_1) \geq 0$ on $(x_1, +\infty)$, then $\overline{V}$ grows at least linearly; and this contradicts the assumption that $\rho_U \geq \underline{\rho}_U > 0$. Overall, then, we must have $\overline{V}'' < 0$ on $(0, +\infty)$. ■

5.4. The Generalized Euler Equation. Since $U'\left(\overline{C}\right)$ may have a discontinuity at 0, we cannot use Itô's Lemma to study its dynamics. We can, however, use Itô's Lemma to study the dynamics of $M = \overline{V}'$. These dynamics are very closely related to those of $U'\left(\overline{C}\right)$. Indeed:

1. if $\overline{C}(0+) = \overline{C}(0)$, then the dynamics of M are identical to those of $U'\left(\overline{C}\right)$;

2. if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) \in (0, +\infty)$, then the dynamics of M are identical to those of $U'\left(\overline{C}\right)$ on the interval $(0, \tau)$, where τ is the first time that x hits 0; and

3. if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) = 0$, then the dynamics of M are identical to those of $U'\left(\overline{C}\right)$, in the sense that both are trivial.

The two dynamics only differ if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) \in (0, +\infty)$, in which case $U'\left(\overline{C}\right)$ jumps up at τ.

Theorem 12. *We have:*

$$\frac{dM}{M} = \left(\gamma - \mu + (1-\alpha)\overline{C}' + \sigma^2 \rho_U\left(\overline{C}\right)\frac{x\overline{C}'}{\overline{C}}\right)dt - \sigma\rho_U\left(\overline{C}\right)\frac{x\overline{C}'}{\overline{C}}dz \qquad (22)$$

if either $x > 0$ or $x = 0$ and $\overline{C}(0+) = \overline{C}(0)$; and

$$\frac{dM}{M} = 0$$

if $x = 0$ and $\overline{C}(0+) > \overline{C}(0)$.

This theorem describes the the exact evolution of 'marginal utility' (specifically $M = \overline{V}'$). The equation includes a stochastic term, (i.e., the final term, which includes dz), and deterministic terms (i.e., the terms which include dt). The stochastic term captures the negative effect that positive wealth shocks have on marginal utility.

The first deterministic term, γdt, implies that marginal utility rises more quickly the higher the long-run discount rate (γ). The $-\mu dt$ term implies that marginal utility rises more slowly the higher the rate of return (μ). The $\sigma^2\rho_U\left(\overline{C}\right)\frac{x\overline{C}'}{\overline{C}}dt$ term captures two separate effects. First, asset income uncertainty (σ^2) affects the savings decision. Second, since marginal utility is non-linear in consumption, asset income uncertainty affects the average value of future marginal utility. The net impact of these two effects is always positive.

The $(1-\alpha)\overline{C}'dt$ term captures the effect of hyperbolic discounting. Naturally, when $\alpha = 1$ this effect vanishes and the model coincides with the standard exponential discounting case. Since $\overline{C}'$ rises as α falls, the hyperbolic effect $(1-\alpha)\overline{C}'$ rises as α falls. Marginal utility rises more quickly the lower the level of α.

Proof. We begin by applying Itô's Lemma to M to obtain

$$dM = \left(\frac{1}{2}\sigma^2 x^2 M'' + \left(\mu x + y - \overline{C}\right) M'\right) dt + \sigma x M' dz. \tag{23}$$

Next, we put $\widetilde{C} = f(\alpha M)$. Then, differentiating equation (14) with respect to x, we have

$$\frac{1}{2}\sigma^2 x^2 M'' + \left(\mu x + y - \widetilde{C}\right) M' - \gamma M + \sigma^2 x M' + \mu M - \widetilde{C}' M + U'\left(\widetilde{C}\right)\widetilde{C}' = 0$$

when $x > 0$. Moreover this equality extends by continuity to the case $x = 0$. Hence

$$\begin{aligned}
\frac{1}{2}\sigma^2 x^2 M'' + \left(\mu x + y - \overline{C}\right) M' &= \frac{1}{2}\sigma^2 x^2 M'' + \left(\mu x + y - \widetilde{C}\right) M' + \left(\widetilde{C} - \overline{C}\right) M' \\
&= \gamma M - \sigma^2 x M' - \mu M + \widetilde{C}' M - U'\left(\widetilde{C}\right)\widetilde{C}' + \left(\widetilde{C} - \overline{C}\right) M' \\
&= \gamma M - \sigma^2 x M' - \mu M + \widetilde{C}' M - \alpha M \widetilde{C}' + \left(\widetilde{C} - \overline{C}\right) M' \\
&= \left(\gamma - \mu + (1-\alpha)\widetilde{C}'\right) M - \left(\sigma^2 x - \left(\widetilde{C} - \overline{C}\right)\right) M'
\end{aligned}$$

and

$$\frac{dM}{M} = \left(\gamma - \mu + (1-\alpha)\widetilde{C}' - \sigma^2 x \frac{M'}{M} + \left(\widetilde{C} - \overline{C}\right)\frac{M'}{M}\right) dt + \sigma x \frac{M'}{M} dz.$$

Next,

$$\frac{M'}{M}=\frac{U''\left(\widetilde{C}\right)\widetilde{C}'}{U'\left(\widetilde{C}\right)}=\frac{\widetilde{C}U''\left(\widetilde{C}\right)}{U'\left(\widetilde{C}\right)}\frac{\widetilde{C}'}{\widetilde{C}}=-\rho_U\left(\widetilde{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}.$$

Hence

$$\frac{dM}{M}=\left(\gamma-\mu+(1-\alpha)\widetilde{C}'+\sigma^2\rho_U\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}}-\left(\widetilde{C}-\overline{C}\right)\rho_U\left(\widetilde{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}\right)dt$$

$$-\sigma\rho_U\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}}dz. \tag{24}$$

In particular, we have the first statement of the Theorem. As for the second statement, note that if $x=0$ and $\overline{C}(0+)>\overline{C}(0)$ then $\overline{C}(0)=y$ and therefore it follows directly from equation (23) that $dM=0$. In particular, we have

$$\left(\widetilde{C}-\overline{C}\right)\rho_U\left(\widetilde{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}=\gamma-\mu+(1-\alpha)\widetilde{C}'.$$

I.e. the correction term in equation (24) exactly cancels the other terms. ■

5.5. The Dynamics of Consumption. Since $\overline{C}$ may have a discontinuity at 0, we cannot use Itô's Lemma to study its dynamics. We can, however, use Itô's Lemma to study the dynamics of $\widetilde{C}=f(\alpha M)$. Just as the dynamics of M were closely related to those of $U'\left(\overline{C}\right)$, so the dynamics of $\widetilde{C}$ are very closely related to those of $\overline{C}$. Indeed:

1. if $\overline{C}(0+)=\overline{C}(0)$, then the dynamics of $\widetilde{C}$ are identical to those of $\overline{C}$;

2. if $\overline{C}(0+)>\overline{C}(0)$ and $x(0)\in(0,+\infty)$, then the dynamics of $\widetilde{C}$ are identical to those of $\overline{C}$ on the interval $(0,\tau)$, where τ is the first time that x hits 0; and

3. if $\overline{C}(0+)>\overline{C}(0)$ and $x(0)\in 0$, then the dynamics of $\widetilde{C}$ are identical to those of $\overline{C}$, in the sense that both are trivial.

The two dynamics only differ if $\overline{C}(0+) > \overline{C}(0)$ and $x(0) \in (0, +\infty)$, in which case $\overline{C}$ jumps down at τ.

For all $c \in (0, +\infty)$, put $\pi_U(c) = -\frac{cU'''(c)}{U''(c)}$, which is Kimball's (1990) prudence coefficient. Then:

Theorem 13. *We have:*

$$\frac{d\widetilde{C}}{\widetilde{C}} = -\left(\frac{\gamma - \mu + (1-\alpha)\widetilde{C}'}{\rho_U\left(\widetilde{C}\right)} + \sigma^2\frac{x\widetilde{C}'}{\widetilde{C}} - \frac{1}{2}\sigma^2\pi_U\left(\widetilde{C}\right)\left(\frac{x\widetilde{C}'}{\widetilde{C}}\right)^2\right)dt + \sigma\frac{x\widetilde{C}'}{\widetilde{C}}dz \quad (25)$$

if either $x > 0$ or $x = 0$ and $\overline{C}(0+) = \overline{C}(0)$; and

$$\frac{d\widetilde{C}}{\widetilde{C}} = 0$$

if $x = 0$ and $\overline{C}(0+) > \overline{C}(0)$.

Equation (25), which describes the evolution of consumption, compares closely to equation (22), which describes the evolution of marginal utility. To underscore the similarities, begin with equation (22), replace $\overline{C}$ with $\widetilde{C}$, and then divide by ρ_U. The only contrasts between this resulting equation and equation (25), are a series of sign reversals and the appearance of the new deterministic term $\frac{1}{2}\sigma^2\pi_U\left(\widetilde{C}\right)\left(\frac{x\widetilde{C}'}{\widetilde{C}}\right)^2 dt$. The sign reversals reflect the inverse relationship between consumption and marginal utility. The new deterministic term reflects the effects of prudence (Kimball, 1990), while the term $\sigma^2\frac{x\widetilde{C}'}{\widetilde{C}}dt$ reflects the impact of risk aversion. The sign of the prudence effect depends on the sign of U'''; when $U''' < 0$ the prudence term raises the growth rate of consumption. By contrast, the sign of the risk aversion effect is always negative.

Proof. We have

$$\frac{d\widetilde{C}}{\widetilde{C}} = \frac{1}{\widetilde{C}}\left(f'(\alpha M)\alpha dM + \frac{1}{2}f''(\alpha M)\alpha^2(dM)^2\right)$$

(applying Itô's Lemma to $\widetilde{C} = f(\alpha M)$)

$$= \frac{1}{\widetilde{C}}\left(f'(\alpha M)\,\alpha M \frac{dM}{M} + \frac{1}{2} f''(\alpha M)\,\alpha^2 M^2 \left(\frac{dM}{M}\right)^2\right)$$

(collecting terms)

$$= \frac{U'\left(\widetilde{C}\right)}{\widetilde{C}U''\left(\widetilde{C}\right)}\frac{dM}{M} - \frac{1}{2}\frac{U'''\left(\widetilde{C}\right)U'\left(\widetilde{C}\right)^2}{\widetilde{C}U''\left(\widetilde{C}\right)^3}\left(\frac{dM}{M}\right)^2$$

(because U' is the inverse of f)

$$= -\frac{1}{\rho_U\left(\widetilde{C}\right)}\frac{dM}{M} + \frac{1}{2}\frac{\pi_U\left(\widetilde{C}\right)}{\rho_U\left(\widetilde{C}\right)^2}\left(\frac{dM}{M}\right)^2$$

(by definition of $\rho_U\left(\widetilde{C}\right)$ and $\pi_U\left(\widetilde{C}\right)$)

$$= -\frac{1}{\rho_U\left(\widetilde{C}\right)}\left(\left(\gamma - \mu + (1-\alpha)\widetilde{C}' + \sigma^2\rho_U\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}} - \left(\widetilde{C} - \overline{C}\right)\rho_U\left(\widetilde{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}\right)dt\right.$$

$$\left. -\sigma\rho_U\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}}dz\right) + \frac{1}{2}\frac{\pi_U\left(\widetilde{C}\right)}{\rho_U\left(\widetilde{C}\right)^2}\left(\sigma\rho_U\left(\widetilde{C}\right)\frac{x\widetilde{C}'}{\widetilde{C}}\right)^2 dt$$

(substituting for $\frac{dM}{M}$ from equation (24))

$$= -\left(\frac{\gamma - \mu + (1-\alpha)\widetilde{C}'}{\rho_U\left(\widetilde{C}\right)} + \sigma^2\frac{x\widetilde{C}'}{\widetilde{C}} - \frac{1}{2}\pi_U\left(\widetilde{C}\right)\left(\sigma\frac{x\widetilde{C}'}{\widetilde{C}}\right)^2 - \left(\widetilde{C} - \overline{C}\right)\frac{\widetilde{C}'}{\widetilde{C}}\right)dt$$

$$+\sigma\frac{x\widetilde{C}'}{\widetilde{C}}dz$$

(simplifying). ■

6. The Deterministic Case

The assumption that $\sigma > 0$ has so far played a crucial, albeit implicit, role in our analysis: it is needed for our derivation of the Bellman system with finite λ, since it allows us to make sense of completely general consumptions functions, including consumption functions that are only measurable; and it is needed for our derivation of the Bellman system of the instantaneous-gratification model, since it allows us to justify the working hypotheses H1-H8. This does not, however, prevent us from studying the limit case of the model obtained when $\sigma \to 0+$. Indeed, doing so is particularly easy in the case of the instantaneous-gratification model, since in this case the hyperbolic problem is equivalent to an appropriately chosen exponential problem.

In the present section we show that, by viewing the deterministic instantaneous-gratification model as a limiting case of the stochastic instantaneous-gratification model in this way, we are able to pinpoint a unique equilibrium value function for the deterministic instantaneous-gratification model. Furthermore, the Bellman system that characterizes this value function turns out to be particularly tractable. In particular, by restricting attention to the case where U has constant relative risk aversion, we provide an example that shows that the condition $\gamma \geq \mu$ used in our proof of monotonicity of the consumption function is necessary, at least in the deterministic case. Assumptions A1-A5 and B1-B3 will be in force throughout the section.

6.1. Derivation of the Deterministic Model. The following theorem describes the sense in which the deterministic instantaneous-gratification model is the limit of the stochastic instantaneous-gratification model. The proof of the theorem, which is beyond the scope of the present paper, is omitted.

Theorem 14. *We have:*

1. *there is a continuous function* $\overline{V}_0 : [0, +\infty) \to \mathbb{R}$ *such that* $\overline{V} \to \overline{V}_0$ *uniformly on compact subsets of* $[0, +\infty)$ *as* $\sigma \to 0+$;

2. $\overline{V}_0$ *is the unique viscosity solution of*

$$(\mu x + y)\,\overline{V}_0' - \gamma \overline{V}_0 + \widehat{h}\left(\overline{V}_0'\right) = 0 \tag{26}$$

when $x > 0$ *and*

$$y\overline{V}_0' - \gamma \overline{V}_0 + \widehat{h}_0\left(\overline{V}_0'\right) = 0 \tag{27}$$

when $x = 0$. ■

In particular, we obtain an equilibrium-refinement result for the deterministic model.

6.2. The Case of Constant Relative Risk Aversion. In this section we adopt the following parametric assumptions:

P1 ρ_U is constant;

P2 $\mu > 0$.

Under these assumptions, we can transform the non-autonomous system (26-27) into an autonomous system.

Lemma 15. *Suppose that assumptions P1-P2 hold, and that* $\rho_U \neq 1$. *Put*

$$l = \log(\mu x + y) \text{ and } v(l) = \frac{\overline{V}_0\left(\frac{\exp(l) - y}{\mu}\right)}{(1 - \rho_U)\,U(\exp(l))}.$$

Then $\overline{V}_0$ *satisfies equations (26-27) iff* v *satisfies*

$$\mu\left((1 - \rho_U)\,v + v'\right) - \gamma v + \widehat{h}\left(\mu\left((1 - \rho_U)\,v + v'\right)\right) = 0 \tag{28}$$

when $l > \log(y)$ *and*

$$\max\left\{\frac{1}{\alpha}, \mu\left((1-\rho_U)v + v'\right)\right\} - \gamma v + \widehat{h}\left(\max\left\{\frac{1}{\alpha}, \mu\left((1-\rho_U)v + v'\right)\right\}\right) = 0 \quad (29)$$

when $l = \log(y)$.

Proof. We proceed in three steps. First, put

$$v_0(x) = \frac{\overline{V}_0(x)}{(1-\rho_U)\,U(\mu x + y)}.$$

Then (26) holds iff

$$0 = (\mu x + y)(1-\rho_U)\left(\mu U' v_0 + U v_0'\right) - \gamma(1-\rho_U)\,U v_0 + \widehat{h}\left((1-\rho_U)\left(\mu U' v_0 + U v_0'\right)\right)$$

(because $\overline{V}_0' = (1-\rho_U)\left(\mu U' v_0 + U v_0'\right)$, and where we have suppressed the dependence of U and v_0 on $\mu x + y$ and x respectively)

$$\Leftrightarrow 0 = (\mu x + y)\left(\frac{\mu U' v_0}{U} + v_0'\right) - \gamma v_0 + \frac{\widehat{h}\left((1-\rho_U)\left(\mu U' v_0 + U v_0'\right)\right)}{(1-\rho_U)\,U}$$

(dividing through by $(1-\rho_U)\,U$)

$$\Leftrightarrow 0 = \left((1-\rho_U)\,\mu v_0 + (\mu x + y)\,v_0'\right) - \gamma v_0 + \widehat{h}\left(\frac{(1-\rho_U)\left(\mu U' v_0 + U v_0'\right)}{\left((1-\rho_U)\,U\right)^{\frac{\rho_U}{\rho_U - 1}}}\right)$$

(because $(\mu x + y)\,U' = (1-\rho_U)\,U$ and $\widehat{h}$ is homogeneous of degree $1 - \frac{1}{\rho_U}$)

$$\Leftrightarrow 0 = \left((1-\rho_U)\,\mu v_0 + (\mu x + y)\,v_0'\right) - \gamma v_0 + \widehat{h}\left((1-\rho_U)\,\mu v_0 + (\mu x + y)\,v_0'\right) \quad (30)$$

(because $((1-\rho_U)\,U)^{\frac{\rho_U}{\rho_U-1}} = U'$). Secondly, put

$$v\,(l) = v_0\left(\frac{\exp\,(l) - y}{\mu}\right).$$

Then (30) holds iff

$$0 = ((1-\rho_U)\,\mu v + \mu v') - \gamma v + \widehat{h}\,((1-\rho_U)\,\mu v + \mu v')$$

(because $v_0'\,(x) = \frac{\mu v'(\log(\mu x+y))}{(\mu x+y)}$). Thirdly, note that

$$\overline{V}_0' \geq \frac{U'\,(y)}{\alpha} \Leftrightarrow ((1-\rho_U)\,\mu v + \mu v') \geq \frac{1}{\alpha}.$$

The same chain of reasoning therefore shows that (27) holds iff

$$0 = \max\left\{\frac{1}{\alpha}, (1-\rho_U)\,\mu v + \mu v'\right\} - \gamma v + \widehat{h}\left(\max\left\{\frac{1}{\alpha}, (1-\rho_U)\,\mu v + \mu v'\right\}\right). \blacksquare$$

Lemma 16. *Suppose that assumptions P1-P2 hold, and that $\rho_U = 1$. Put*

$$l = \log\,(\mu x + y) \ \text{ and } v\,(l) = \overline{V}_0\left(\frac{\exp\,(l) - y}{\mu}\right) - \frac{U\,(\exp\,(l))}{\gamma}.$$

Then $\overline{V}_0$ satisfies equations (26-27) iff v satisfies

$$\mu\left(\frac{1}{\gamma} + v'\right) - \gamma v + \widehat{h}\left(\mu\left(\frac{1}{\gamma} + v'\right)\right) = 0 \tag{31}$$

when $l > \log\,(y)$ and

$$\max\left\{\frac{1}{\alpha}, \mu\left(\frac{1}{\gamma} + v'\right)\right\} - \gamma v + \widehat{h}\left(\max\left\{\frac{1}{\alpha}, \mu\left(\frac{1}{\gamma} + v'\right)\right\}\right) = 0 \tag{32}$$

when $l = \log\,(y)$.

Proof. We proceed in three steps. First, put

$$v_0(x) = \overline{V}_0(x) - \frac{U(\mu x + y)}{\gamma}.$$

Then (26) holds iff

$$(\mu x + y)\overline{V}_0' - \gamma \overline{V}_0 + \widehat{h}\left(\overline{V}_0'\right) = 0$$

$$0 = (\mu x + y)\left(\frac{\mu U'}{\gamma} + v_0'\right) - \gamma\left(\frac{U}{\gamma} + v_0\right) + \widehat{h}\left(\frac{\mu U'}{\gamma} + v_0'\right)$$

(because $\overline{V}_0' = \frac{\mu U'}{\gamma} + v_0'$, and where we have suppressed the dependence of U and v_0 on $\mu x + y$ and x respectively)

$$\Leftrightarrow 0 = \left(\frac{\mu}{\gamma} + (\mu x + y) v_0'\right) - \gamma\left(\frac{U}{\gamma} + v_0\right) + \widehat{h}\left(\frac{\frac{\mu}{\gamma} + (\mu x + y) v_0'}{(\mu x + y)}\right)$$

(because $(\mu x + y) U' = 1$)

$$\Leftrightarrow 0 = \left(\frac{\mu}{\gamma} + (\mu x + y) v_0'\right) - \gamma v_0 + \widehat{h}\left(\frac{\mu}{\gamma} + (\mu x + y) v_0'\right) \tag{33}$$

(because $\widehat{h}(\phi) = -\log(\alpha\phi) - \frac{1}{\alpha}$ and $U = \log(\mu x + y)$). Secondly, put

$$v(l) = v_0\left(\frac{\exp(l) - y}{\mu}\right).$$

Then (33) holds iff

$$0 = \left(\frac{\mu}{\gamma} + \mu v'\right) - \gamma v + \widehat{h}\left(\frac{\mu}{\gamma} + \mu v'\right)$$

(because $v_0'(x) = \frac{\mu v'(\log(\mu x + y))}{(\mu x + y)}$). Thirdly, note that

$$\overline{V}_0' \geq \frac{U'(y)}{\alpha} \Leftrightarrow \left(\frac{\mu}{\gamma} + \mu v'\right) \geq \frac{1}{\alpha}.$$

The same chain of reasoning therefore shows that (27) holds iff

$$0 = \max\left\{\frac{1}{\alpha}, \frac{\mu}{\gamma} + \mu v'\right\} - \gamma v + \widehat{h}\left(\max\left\{\frac{1}{\alpha}, \frac{\mu}{\gamma} + \mu v'\right\}\right). \blacksquare$$

Theorem 17. *Suppose that assumptions P1-P2 hold. Then:*

1. *If $\mu \leq \gamma < +\infty$, then $\overline{C}' > 0$ on $(0, +\infty)$.*
2. *If $\alpha\mu < \gamma < \mu$, then there exists $x_1 \in (0, +\infty)$ such that:*
   (a) *$\overline{C}' > 0$ on $(0, x_1)$;*
   (b) *$\overline{C}' > 0$ and $\overline{C}'' = 0$ on $(x_1, +\infty)$; and*
   (c) *$\overline{C}(x_1+) < \overline{C}(x_1-)$.*
3. *If $\gamma \leq \alpha\mu$, then $\overline{C}' > 0$ and $\overline{C}'' = 0$ on $(0, +\infty)$.*

Moreover, if $\gamma > \alpha\mu$, then $\overline{C}(x) > \mu x + y$ for all $x \in [0, +\infty)$. In particular, there is always a unique solution to the wealth dynamics.

Proof. Put

$$a = \left\{\begin{array}{ll} \frac{1}{(1-\rho_U)\gamma} & \text{if } \rho_U \neq 1 \\ 0 & \text{if } \rho_U = 1 \end{array}\right\}.$$

Then, in view of Lemmas 15 and 16, there exists a smooth function $H : (-\infty, +\infty) \to \mathbb{R}$ such that:

1. $v = H(v')$;
2. $H'' > 0$;
3. $H(\phi) \to +\infty$ as $\phi \to \pm\infty$;
4. $\min H < a$; and

5. if $H'(0) \leq 0$ then $H(0) < a$.

Moreover, because v is a viscosity solution of the equation $v = H(v')$, any switches that occur between the two values of v' consistent with any given v must be from the lower to the higher value of v'. There are therefore three possibilities:

1. If $H'(0) \leq 0$, then: $v(0) = a$; $v' < 0$ on $(0, +\infty)$; and v asymptotes to $H(0)$.
2. If $H'(0) > 0$ and $H(0) < a$, then there exists $x_1 \in (0, +\infty)$ such that: $v(0) = a$; $v' < 0$ on $(0, x_1)$; and $v = H(0)$ on $[x_1, +\infty)$. In particular, v' jumps from the lower to the higher of the two values in $H^{-1}(H(0))$ at x_1.
3. If $H(0) \geq a$, then: $v = H(0)$ on $[0, +\infty)$.

Moreover, it can be shown that

$$H'(0) = -\frac{\gamma - \mu}{\gamma - \mu(1 - \rho_U)},$$

and that

$$\max H^{-1}(H(a)) = \frac{\gamma - \alpha\mu}{\alpha\mu\gamma}.$$

Finally, it can be shown that $\overline{C}(x) > \mu x + y$ iff $v'(l) < \frac{\gamma - \alpha\mu}{\alpha\mu\gamma}$. Hence $\overline{C}(x) > \mu x + y$ for all $x \in [0, +\infty)$ iff $H(0) < a$. In particular, while $\overline{C}$ fails to be unique at x_1 in the second of our three cases, there is nonetheless a unique solution to the dynamics even in that case. ■

7. The Bellman System of the Instantaneous-Gratification Model Revisited

The derivation of the Bellman system of the instantaneous-gratification model given in Section 3 is unsatisfactory in that it relies on Hypotheses H1-H8, which involve entities that are endogenous to the model. It is therefore reassuring to know that these hypotheses can be established using Assumptions A1-A5 and B1-B3. In particular, we have the following theorem. The proof of the theorem, which is beyond the scope of the present paper, is omitted.

Theorem 18. *Suppose that Assumptions A1-A5 and B1-B3 hold. Then:*

1. W *and* V *are continuous on* $[0, +\infty)$*;*
2. *there exist continuous functions* $\overline{W}, \overline{V} : [0, +\infty) \to \mathbb{R}$ *such that* $W \to \overline{W}$ *and* $V \to \overline{V}$ *uniformly on compact subsets of* $[0, +\infty)$ *as* $\lambda \to +\infty$*;*
3. $\overline{W} = \alpha \overline{V}$*;*
4. $\overline{V}$ *is the unique viscosity solution of*

$$\frac{1}{2}\sigma^2 x^2 \overline{V}'' + (\mu x + y)\,\overline{V}' - \gamma \overline{V} + \widehat{h}\left(\overline{V}'\right) = 0$$

when $x > 0$ *and*

$$y\overline{V}' - \gamma \overline{V} + \widehat{h}_0\left(\overline{V}'\right) = 0$$

when $x = 0$. ■

Remark 19. *Theorem 18 covers the case in which* U *has constant relative risk aversion* $\rho_U > 1 - \alpha$.

A satisfactory limit theorem can also be proved under the following assumptions, which complement Assumptions B2 and B3:

B2′ $\alpha + \overline{\rho}_U - 1 < 0$;

B3′ $(2 - \alpha) - (1 - \alpha)\,\underline{\theta}_U < 0$.

Indeed, we have the following theorem. The proof of the theorem, which is beyond the scope of the present paper, is omitted.

Theorem 20. *Suppose that Assumptions A1-A5, B1, B2′ and B3′ hold. Then:*

1. *W and V are continuous on $[0, +\infty)$;*
2. *$W \to \frac{\alpha U(y)}{\gamma}$ and $V \to \frac{U(y)}{\gamma}$ uniformly on compact subsets of $[0, +\infty)$ as $\lambda \to +\infty$.* ■

These limiting value functions arise because, as $\lambda \to +\infty$, the option of consuming here entire wealth becomes attractive to the current self. In effect, her utility function is not sufficiently concave.

Remark 21. *Theorem 20 covers the case in which U has constant relative risk aversion $\rho_U < 1 - \alpha$.*

Comparing Assumptions B2 and B3 with Assumptions B2′ and B3′, it is clear that there is a knife-edge case in between, namely the case in which U has constant relative risk aversion $\rho_U = 1 - \alpha$. We have not analyzed this case. However, we would expect it to resemble the case $\rho_U > 1 - \alpha$ covered by Theorem 20.

Finally, note that Theorem 18 continues to hold when Assumptions B2 and B3 are replaced by the following, significantly weaker, assumptions:

B2″ $\alpha + \left(\liminf_{c \to +\infty} \frac{-cU''(c)}{U'(c)^2}\right) - 1 > 0$;

B3″ $(2 - \alpha) - (1 - \alpha)\left(\limsup_{c \to +\infty} \frac{U'(c)U'''(c)}{U''(c)^2}\right) > 0$.

Assumptions B2″ and B3″ ensure that $\widehat{h}$ is decreasing and convex near 0. This is enough to ensure that consumption remains bounded as $\lambda \to +\infty$. These assumptions are, however, consistent with $\widehat{h}$ being increasing or concave away from 0. In other words, for some BRRA utility functions, the instantaneous-gratification problem is not value-function equivalent to any exponential consumption problem.

8. Conclusions

We have described a continuous-time model of hyperbolic discounting. Our model allows for a general class of preferences, includes liquidity constraints, and places no restrictions on equilibrium policy functions. The model is also psychologically relevant. We take the phrase "instantaneous gratification" literally. We analyze a model in which individuals prefer gratification in the present instant discretely more than consumption in the momentarily delayed future. In this simple setting, equilibrium is unique and the consumption function is continuous. When the long-run discount rate weakly exceeds the interest rate, the consumption function is also monotonic. All of the pathologies that characterize discrete-time hyperbolic models vanish.

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