INSTANTANEOUS GRATIFICATION

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Abstract. We propose a tractable continuous-time model that captures the key psychological properties of the discrete-time quasi-hyperbolic discount function. Like the discrete-time model, our 'instantaneous-gratification' model reflects consumers' preference to act impatiently in the short run and patiently in the long run. Unlike the discrete-time model, the instantaneous-gratification model generates policy functions that are continuous and monotonic, admits only one equilibrium, and features a single welfare criterion. We illustrate these useful properties using a standard consumption model with liquidity constraints. The instantaneous-gratification model eliminates the problematic and counterfactual properties of the discrete-time hyperbolic model, but preserves the model's desirable psychological features.

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1. Introduction

Everyday speech has many terms that describe the drive for immediate rewards. We say that consumers seek 'instant gratification' and 'immediate gratification'; that they 'live only for the moment'; or that they 'want it now.'

Robert Strotz (1956) was the first to model instant gratification mathematically. He pointed out that two ingredients are essential to a successful theory. First, the discount function should depend on the difference between the current time and the future time at which the discounted reward is consumed. Second, the discount function should not be exponential.[1] He went on to conjecture that empirical discount rates would decrease with the time horizon.[2] In other words, delaying current consumption by one period produces proportionately more devaluation than delaying future consumption by one period. Most experimental studies of time preference have supported Strotz's conjecture (Ainslie 1992, Loewenstein and Read 2001), although debate continues about the shape and even the existence of a single discount function (Frederick, Loewenstein and O'Donoghue 2002).

To parameterize these discounting properties and the taste for instant gratification, Laibson (1997a) adopted the discrete-time discount function $\{1, \beta\,\delta, \beta\,\delta^2, \beta\,\delta^3, ...\}$ — which Phelps and Pollak (1968) had previously used to model intergenerational time preferences. With $\beta < 1$, this so called 'quasi-hyperbolic' discount function generates a gap between a high short-run discount rate and a low long-run rate. O'Donoghue and Rabin (1999a, 1999b) call these 'present-biased' time preferences, emphasizing the heightened weight they place on current consumption. In the last several years, the quasi-hyperbolic discount function has been used to study a wide range of behaviors, including consumption, procrastination, addiction and job search.[3]

The quasi-hyperbolic discounting model has at least three significant drawbacks. First, it generates multiple equilibria, raising questions about its empirical usefulness.[4] A model that cannot be pinned down to a single equilibrium prediction is hard to falsify. Second, it generates counterfactual policy functions. Consumption functions in quasi-hyperbolic

[1] Strotz (1956, p. 165 and Section V).

[2] Strotz (1956, p. 177) states that special attention should be given to discount functions that depart from the exponential case by overvaluing "the more proximate satisfactions relative to the more distant ones". Moreover footnote 1 on the same page makes clear that this statement applies to discount rates. Figure 3 on p. 175 is misleading. (It depicts a discount function for which discount rates are initially *increasing* in the time difference.)

[3] For some examples, see Angeletos, Laibson, Repetto, Tobacman and Weinberg (2001), O'Donoghue and Rabin (1999a, 1999b), Della Vigna and Paserman (2000), DellaVigna and Malmendier (2004).

[4] See Krusell and Smith (2000) for a proof of non-uniqueness.

models need not be globally monotonic in wealth, and may even drop discontinuously at a countable number of points.[5] Figure 1 plots examples of such 'pathological' consumption functions from discrete-time models. Third, it does not generate an obvious welfare criterion, since different selves have conflicting preferences.

The current paper shows how to model the taste for instant gratification in continuous time, and shows that the continuous-time model has a natural limit case that eliminates all of the problems summarized above.[6]

The *general* version of our continuous-time model captures the qualitative properties of the original discrete-time quasi-hyperbolic model. It makes a clear distinction between the 'present' and the 'future', a psychological contrast supported by recent fMRI brain-imaging evidence.[7] We assume that the present is valued discretely more than the future, mirroring the one-time drop in valuation implied by the discrete-time quasi-hyperbolic discount function (Phelps and Pollak 1998, Laibson 1997) and its continuous-time generalizations (Barro 1999, Luttmer and Mariotti 2000). We also assume that the transition from the present to the future is determined by a constant hazard rate. This simplifying assumption enables us to reduce the Bellman equation for our problem to a system of two *stationary* ordinary differential equations that characterize present and future value functions.

The *limit* version of our continuous-time model is derived by making the present vanishingly short. This version is analytically tractable and psychologically relevant. By focusing on this psychologically important case, we take the phrase 'instant gratification' literally: in our model, individuals prefer gratification in the present instant discretely more than consumption in the only slightly delayed future. Hence, the limit case reflects sharp short-run discounting, a pattern of behavior that has been documented in laboratory experiments.[8]

We call the limit version of our model the instantaneous-gratification model, or IG model

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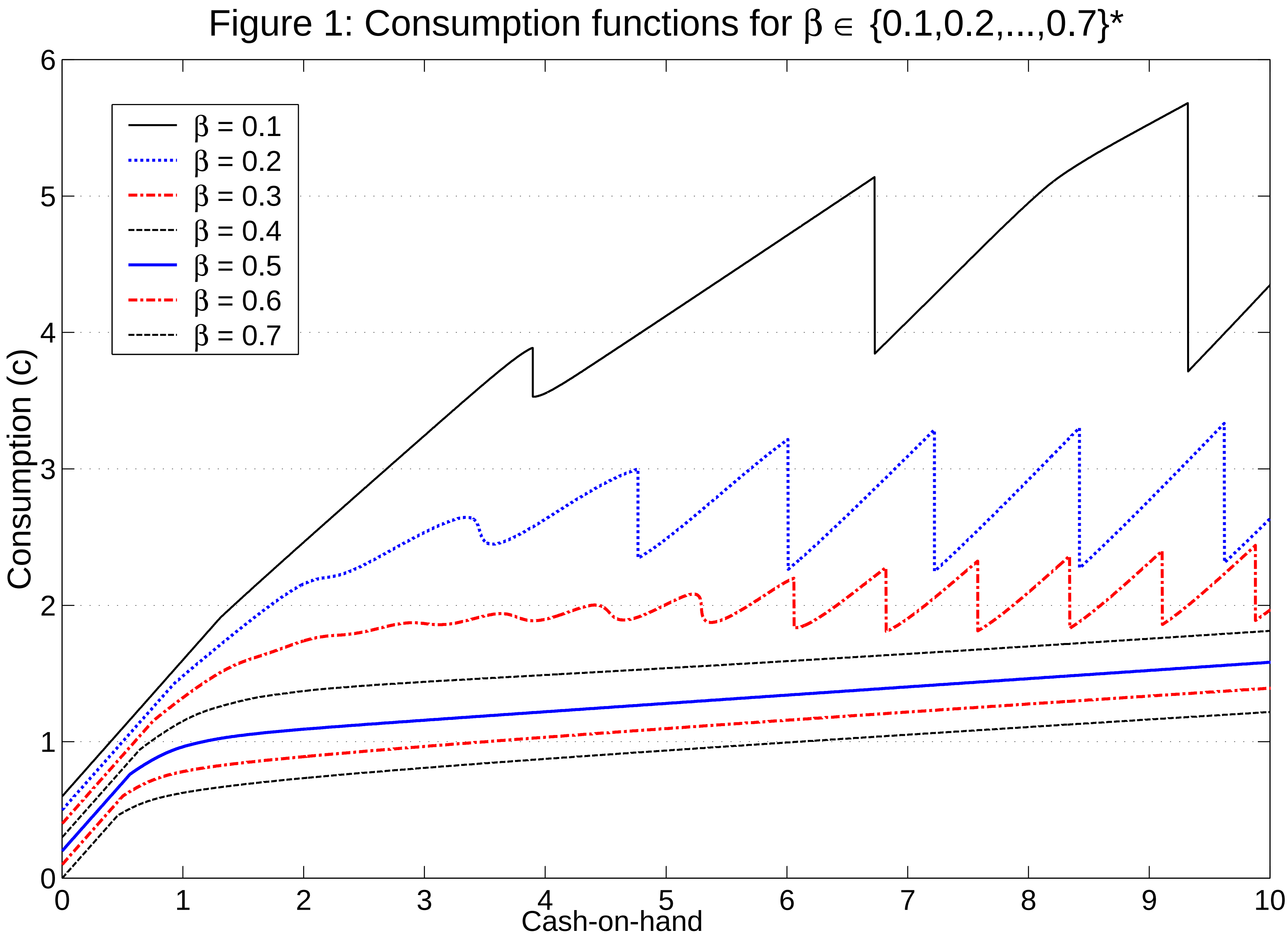
[5] See Laibson (1997b), Morris and Postlewaite (1997), O'Donoghue and Rabin (1999a), Harris and Laibson (2001b), and Krusell and Smith (2000).

[6] Two other solutions to the first three of these problem have been proposed. First, Harris and Laibson (2001b) point out that pathologies occur only in a limited region of the parameter space (notably when the coefficient of relative risk aversion lies well below unity and when β is sufficiently far below unity). Second, O'Donoghue and Rabin (1999a) point out that pathologies do not arise if consumers naively believe that their preferences are dynamically consistent. However, even partial knowledge of future dynamic inconsistency reinstates the pathologies.

[7] McClure, Laibson, Loewenstein, and Cohen (2004) find that the limbic and para-limbic cortical systems are activated when subjects evaluate immediate rewards and not when subjects evaluate delayed rewards. This implies that the emotional/affective (i.e., limbic) system only makes a distinction between present and future rewards, instead of showing a gradual gradient with respect to time delay.

[8] Add citations here. Michel et al (1975), Ainslie (1991), etc...



*These consumption functions are taken from discrete time simulations in Harris and Laibson (2001b). These simulations assume iid income, a risk-free asset, and CRRA. The short-run discount factor is $\beta\delta$. The long-run discount factor is δ=.95. The plotted consumption functions are shifted upward (in increments of .1) so they do not overlap.

for short. We show that the IG model, which is dynamically *inconsistent*, shares the same value function as a related dynamically-*consistent* optimization problem with a wealth-contingent utility function. Using this partial equivalence, we can show both existence and uniqueness of the IG equilibrium. However, our model is not observationally equivalent to the related dynamically consistent optimization problem: the partial equivalence applies to the value functions but *not* to the policy functions.

We also show that the equilibrium consumption function of the IG model is continuous and monotonic in wealth. The monotonicity property relies on the condition that the long-run discount rate is weakly greater than the interest rate.

The IG model has these superior regularity properties — i.e., well-behaved policy functions and uniqueness of equilibrium — because the IG model carves out a special niche between dynamically-inconsistent models and dynamically-consistent models. The IG model features dynamically-inconsistent behavior and rational expectations. So each moment the individual acts strategically with regard to her future preferences. Nevertheless, the fact that the IG value function coincides with the value function of a related dynamically consistent optimization problem, implies that the IG problem inherits many standard regularity properties.

The IG model also features a *single* welfare criterion even though the model generates dynamically inconsistent behavioral choices. Because the present is valued discretely more than the future, the current self has an incentive to overconsume; but the discretely higher value of the present only lasts for an instant, so this overvaluation doesn't affect the welfare criterion. Hence, the model simultaneously features a single welfare function and a behavioral tendency toward overconsumption.

In summary, we argue that the continuous-time IG model is superior to the discrete-time quasi-hyperbolic model. The IG model is more tractable, makes more sensible predictions, supports a unique equilibrium, and identifies a single, sensible welfare criterion.

Two other sets of authors have analyzed quasi-hyperbolic preferences in continuous time. Barro (1999) analyzes the choices of quasi-hyperbolic agents with constant relative risk aversion. He focuses on the general-equilibrium implications of quasi-hyperbolic discounting and the ways in which quasi-hyperbolic economies may be observationally equivalent to exponential economies. Luttmer and Mariotti (2003) analyze the choices of agents with arbitrary discount functions, constant relative risk aversion, and stochastic asset returns. Luttmer-Mariotti generalize Barro's observational-equivalence result, but also identify particular endowment processes for which the quasi-hyperbolic model has interesting new asset-pricing implications (e.g., an elevated equity premium). Luttmer and Mariotti work with general discount functions and consider numerous special cases. They have identified some properties of the particular case in which the present is vanishingly short. However, their findings do not overlap with ours.

Barro and Luttmer-Mariotti both restrict their analysis to *linear* policy rules. The existence of a linear equilibrium depends on special preference assumptions (constant relative risk aversion) and market assumptions (e.g., no liquidity constraints). We do not make restrictive assumptions of this kind: we work with a broad class of preferences; and we introduce a market imperfection: liquidity constraints. We pursue these generalizations for greater realism. Our problem does not admit a linear equilibrium. We have to contend with the problems that arise in our general setting, but do not arise under the Barro/Luttmer-Mariotti simplifying assumptions in either discrete or continuous time.

Our results also differ from Barro and Luttmer-Mariotti in that we are able to prove uniqueness of Markov equilibrium in the class of *all* policy rules. This is a desirable and unexpected result, since the hyperbolic model is a dynamic game, and can therefore generate non-uniqueness. Indeed, Krusell and Smith (2000) have shown that quasi-hyperbolic Markov equilibria are *not* unique in a deterministic discrete-time setting. In the current paper, we provide two uniqueness results. First, we prove uniqueness in the case in which asset returns are stochastic. Second, we propose a refinement that uses the unique equilibrium of the stochastic setting to select a unique sensible equilibrium in the deterministic setting. This refinement takes the natural approach of selecting the limiting equilibrium obtained as the noise in the asset returns vanishes.

The rest of the paper formalizes these claims. In Section 2 we present our general continuous-time model and formulate some of the properties of this model. In Section 3 we present the consumption model that will provide the principal application of the paper. In Section 4 we describe an important limit case of our model. We call this limit case the Instantaneous-Gratification (IG) model. In Section 5 we show that the IG model has the same *Bellman equation* as a related dynamically-consistent optimization problem. However, note that the IG problem is *not* observationally equivalent to the dynamically-consistent optimization problem. The two problems share the same long-run discount rate and the same value function, but they have different instantaneous utility functions and different equilibrium policy functions.[9] In Section 7, we use our partial equivalence result to derive

[9] By contrast, see Barro (1999), Laibson (1996) and Luttmer and Mariotti (2000) for the special case (namely log utility and no liquidity constraints) in which observational equivalence of the policy functions

several important properties of the IG problem, including equilibrium existence, equilibrium uniqueness, consumption-function continuity, and consumption-function monotonicity. In Section 7 we also derive the deterministic version of the IG model, and provide a complete analysis of the case of constant relative risk aversion. In Section 8 we further generalize our results, and in Section 9 we conclude.

2. Time Preferences

2.1. The Basic Model of Time Preferences. In the standard *discrete-time* formulation of quasi-hyperbolic time preferences, it is natural to divide time into two intervals: the present, which consists of the current period only; and the future, which consists of all subsequent periods. All periods, present and future, are discounted exponentially with the discount factor $0 < \delta < 1$. Furthermore, future periods are discounted by the additional factor $0 < \beta \leq 1$. Overall, the present period is discounted with the discount factor 1, and a period $n \geq 1$ steps into the future is discounted with the discount factor $\beta\,\delta^n$ (Phelps and Pollak 1968, Laibson 1997).

This model can be generalized in two ways. First, instead of the present lasting for exactly one period, it can last for an arbitrary length of time. Second, instead of the duration of the present being deterministic, it can be random. Moreover the generalized model has a natural continuous-time analogue.

Consider an economic self born at date s_0. The preferences of this self are divided into two intervals: a 'present', which lasts from s_0 to $s_0 + \tau_0$; and a 'future', which lasts from date $s_0 + \tau_0$ to ∞. Think of the present as the interval during which control is exercised by the current self, and of the future as the interval during which control is exercised by subsequent selves. The length τ_0 of the present is stochastic, and is distributed exponentially with parameter $\lambda \in [0, \infty)$. That is, λ is the hazard rate of the transition from the present to the future.

When the future of self s_0 commences at $s_0 + \tau_0$, a new self is born and takes control of decision-making. Call this self $s_1 = s_0 + \tau_0$. The preferences of this new self can also be divided into two intervals. Self s_1 has a present that lasts from date s_1 to date $s_1 + \tau_1$, and a future that lasts from $s_1 + \tau_1$ to ∞. Extending this idea, we assume that at each juncture of present and future a new self is born, yielding a sequence of selves born at dates $\{s_0, s_1, s_2, ...\}$, with respective present intervals of duration $\{\tau_0, \tau_1, \tau_2, ...\}$.

We assume that all selves discount exponentially with discount factor $0 < \delta < 1$. Fur-

does hold.

thermore, they value their future discretely less than their present, discounting their future by the additional factor $0 < \beta \leq 1$. For example, consider the self that is born at date 0, and which has a present of duration τ. Because the transition date τ is stochastic, self 0 has a stochastic discount function,

$$D(t) = \left\{ \begin{array}{lll} \delta^t & \text{if} & t \in [0, \tau) \\ \beta\, \delta^t & \text{if} & t \in [\tau, \infty) \end{array} \right\}. \tag{1}$$

$D(t)$ decays exponentially at rate $\gamma = -\ln(\delta)$ up to time τ, drops discontinuously at τ to a fraction β of its level just prior to τ, and decays exponentially at rate γ thereafter. Hence, self 0 discounts all flows in her future — i.e., flows that come after time τ — by the extra factor β.

This example illustrates the intertemporal preferences of the self born at date 0. More generally, the formula for $D(t)$ in equation (1) represents the discount factor that self s applies to utility flows that arrive t periods in the future.

This continuous-time formalization is close to some of the deterministic discount functions used in Barro (1999) and Luttmer and Mariotti (2003). However, we assume that τ is stochastic. Figure 2 plots a single realization of this discount function, with $\tau = 3.4$.

As $\lambda \rightarrow 0$, our discount function reduces to the standard exponential discount function:

$$\lim_{\lambda \rightarrow 0} D(t) = \delta^t \text{ for all } t \in [0, \infty).$$

As $\lambda \rightarrow \infty$, the discount function converges to a deterministic jump function with a jump at $t = 0$:

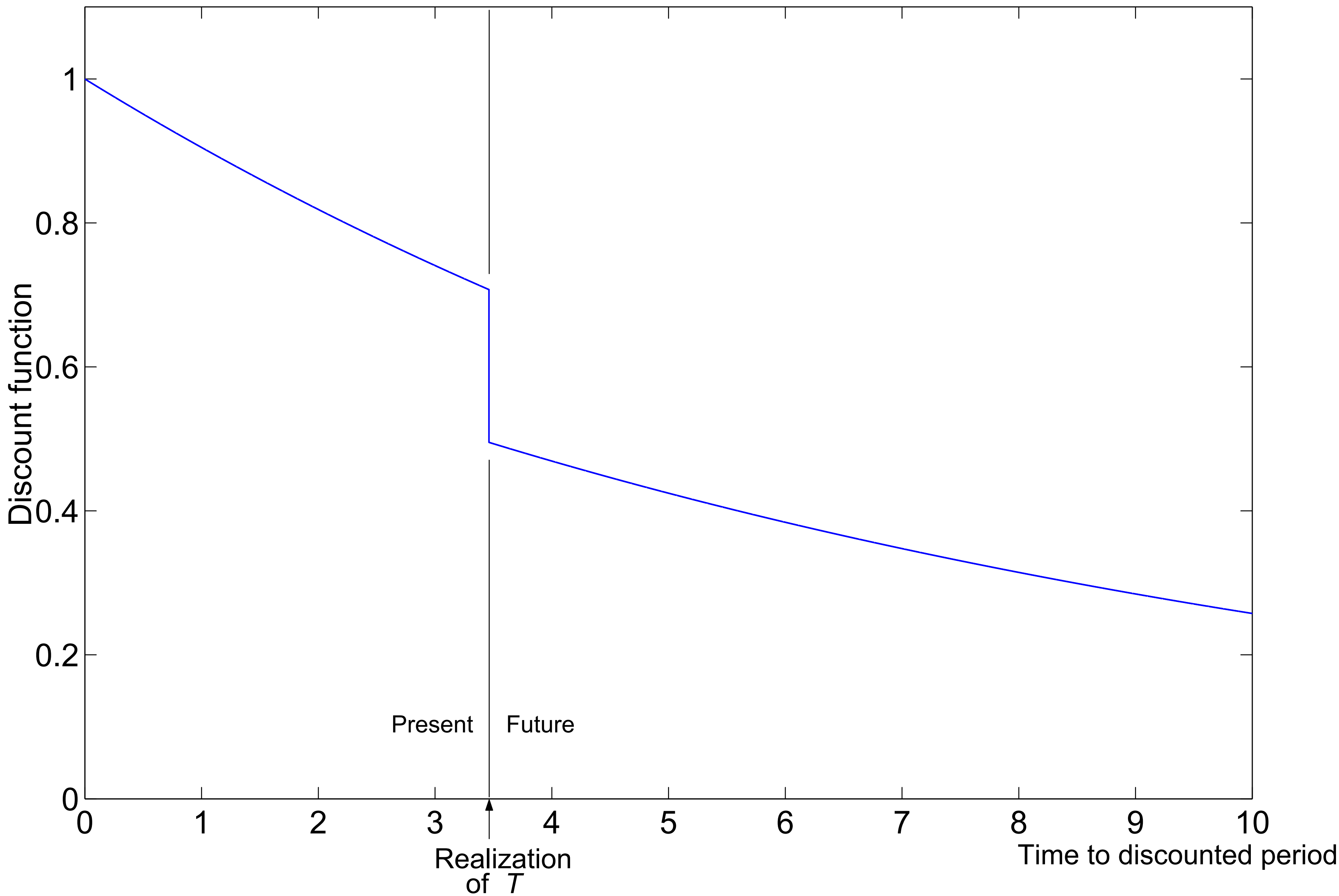
$$\lim_{\lambda \rightarrow \infty} D(t) = \left\{ \begin{array}{lll} 1 & \text{if} & t = 0 \\ \beta\, \delta^t & \text{if} & t \in (0, \infty) \end{array} \right\}.$$

We shall return to the latter case below.

2.2. An Alternative Model of Time Preferences. The arguments in this paper are consistent with a second interpretation of the time preferences described above. In particular, one can assume that a new self is born every *instant*, and that each self has a *deterministic* discount function equal to the expected value of the *stochastic* discount function described above. Recalling that $\gamma = -\ln \delta$, the expectation of the stochastic discount function $D(t)$ is given by

$$\overline{D}(t) = \mathrm{E}\big[D(t)\big] = e^{-\lambda t}\, e^{-\gamma t} + (1 - e^{-\lambda t})\, \beta\, e^{-\gamma t}.$$

Figure 2: Realization of discount function (β=0.7, γ=0.1)



In other words, $\overline{D}(t)$ is the sum of two terms. The first term is the probability $e^{-\lambda t}$ with which the drop in D occurs after time t, times the discount factor $e^{-\gamma t}$ that applies prior to the drop. The second term is the probability $1 - e^{-\lambda t}$ with which the drop in D occurs before time t, times the discount factor $\beta\, e^{-\gamma t}$ that applies after the drop.

The instantaneous discount rate associated with the deterministic discount function $\overline{D}$ is then

$$-\frac{\overline{D}'(t)}{\overline{D}(t)} = \gamma + \frac{\lambda\, e^{-\lambda t}\ (1-\beta)\ e^{-\gamma t}}{\overline{D}(t)}.$$

In other words, the instantaneous discount rate is the sum of two terms. The first term is just the long-run (exponential) discount rate γ. The second term is the ratio of the expected drop in D at time t to the level of $\overline{D}$ at time t. Indeed: $\lambda\, e^{-\lambda t}$ is the flow probability with which the drop in D occurs at time t; and $(1-\beta)\ e^{-\gamma t}$ is the size of the drop in D if the drop occurs at t.

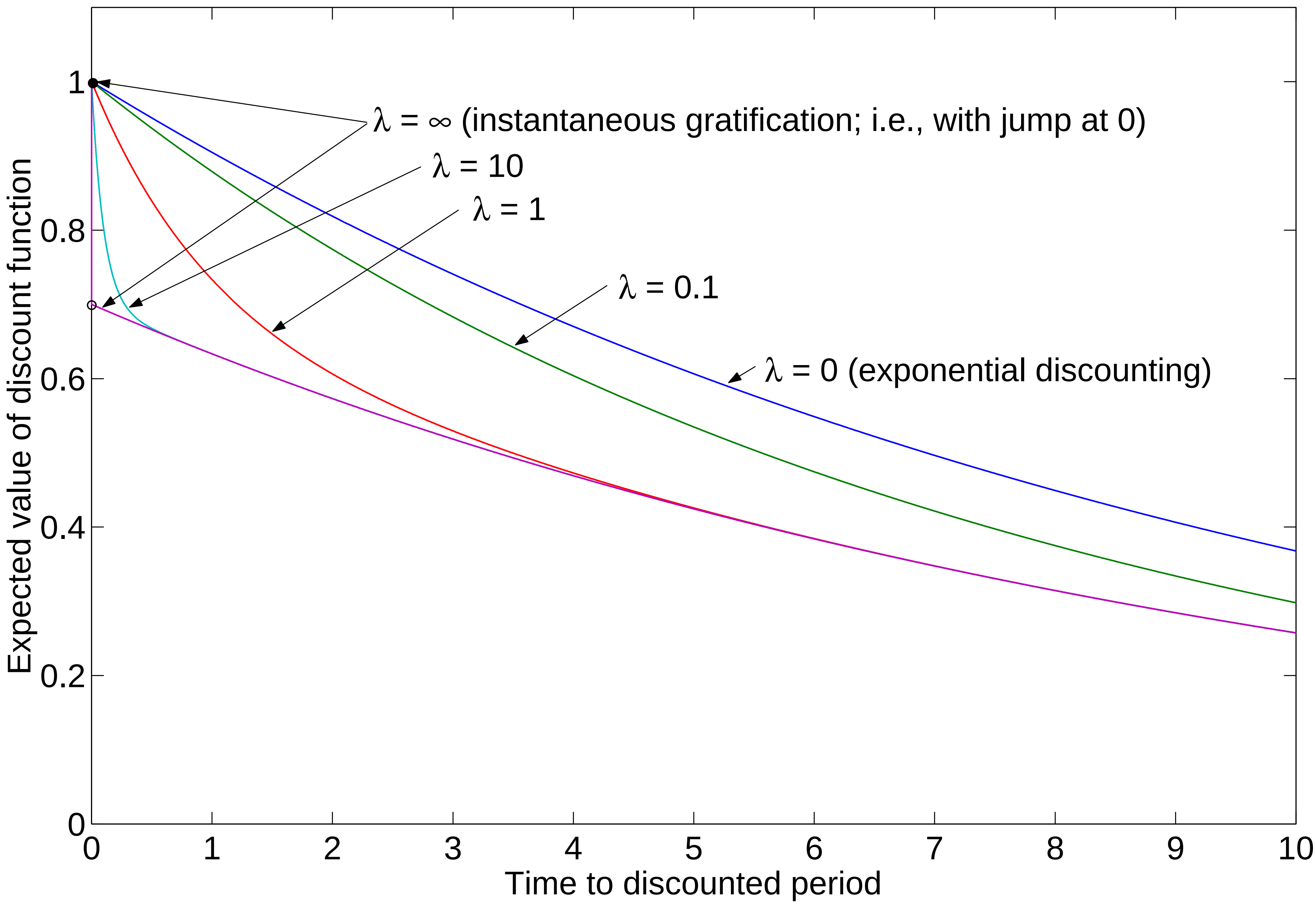
Notice that the instantaneous discount rate decreases from $\gamma + \lambda\ (1-\beta)$ at $t = 0$ to γ at $t = \infty$. Figure 3 plots $\overline{D}(t)$ for $\lambda \in \{0, 0.1, 1, 10, \infty\}$.

2.3. Comparison of the Two Models. At first sight, the basic and alternative models described in subsections 2.1 and 2.2 are quite distinct. After all, the basic model uses a stochastic discount function with a present of non-infinitesimal duration τ, while the alternative model uses a deterministic discount function with a present of infinitesimal duration dt. The basic model involves a countable number of non-infinitesimal selves, while the alternative model involves a continuum of infinitesimal selves. The two models are, however, equivalent.

To see why, note that the current self s in the basic model is dynamically consistent. It therefore makes no difference whether we regard her as a single non-infinitesimal agent, which decides once and for all at the outset of the interval $[s, s+\tau)$ how it will behave throughout this interval, or whether we regard her as a continuum of infinitesimal agents, each of which decides how it will behave at the instant $t \in [s, s+\tau)$ at which it acts. Moreover, if we regard the current self as a continuum of infinitesimal agents, and if we assume that τ is independent of the other stochastic elements of the model, then we can take expectations conditional on those other stochastic elements to conclude that the preferences of the infinitesimal agents of the non-infinitesimal selves of the basic model coincide with the preferences of the infinitesimal selves of the alternative model.

The basic model has two advantages over the alternative model. First, it can be set up using only standard ingredients. Second, in order to analyze this model, we only have to

Figure 3: Expected value of discount function for $\lambda \in \{0,0.1,1,10,\infty\}$



take one limit, namely that obtained as λ goes to infinity. In doing so, we simultaneously pass from non-infinitesimal to infinitesimal selves and from the finite-λ discount function to the infinite-λ discount function that is the ultimate focus of the paper. By contrast, in order to set up the alternative model, we would first have to formalize the idea of an infinitesimal self. This would involve taking the limit as the span of control of a non-infinitesimal self goes to zero. We would then have to let λ go to infinity, in order to pass from the finite-λ discount function to the infinite-λ discount function.

We therefore focus on the basic model in this paper. It should, however, be borne in mind that the alternative model is ultimately the more general model.

3. A Continuous-Time Consumption Model

Two important qualitative features of consumers' planning problems are liquidity constraints and labor-income uncertainty. Cf. Deaton (1991) and Carroll (1992, 1997). We include liquidity constraints in our consumption model, since they make an important difference to the analysis. We exclude labor-income uncertainty, since it complicates the notation and does not affect our conclusions.

3.1. The Dynamics. At any given point in time $t \in [0, \infty)$, the consumer has stock of wealth $x \in [0, \infty)$ and receives a flow of labor income $y \in (0, \infty)$. If $x > 0$ then the consumer is not liquidity constrained, and she may choose any consumption level $c \in (0, \infty)$. Indeed, wealth is a stock and consumption is a flow. Any finite consumption level is therefore achievable provided that it is not maintained for too long. If $x = 0$ then the consumer is liquidity constrained, and she may only choose a consumption level $c \in (0, y]$. Indeed, she has no wealth and she cannot borrow. She cannot therefore consume more than her labor income.

Whatever the consumer does not consume is invested in an asset, the returns on which are distributed normally with mean $\mu\, dt$ and variance $\sigma^2\, dt$, where $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$. The change in her wealth at time t is therefore

$$dx = (\mu\, x + y - c)\, dt + \sigma\, x\, dz,$$

where z is a standard Wiener process.

We could easily generalize this framework by adding a stochastic source of labor income. For example, we could assume that — in addition to her basic flow of labor income y — the agent sporadically receives lump-sum bonuses. To preserve stationarity, such bonuses would need to arrive with a constant hazard rate and be drawn from a fixed distribution. We could even allow for non-stationary labor income, at the expense of an extra state variable. We do not pursue these generalizations, since they would not qualitatively change the analysis that follows.

3.2. Preferences. As discussed above, the consumer is modeled as a sequence of autonomous selves. Each self controls consumption in the present and cares about — but does not directly control — consumption in the future. Now suppose that $c : [0,\infty) \to (0,\infty)$ is a stationary consumption function which takes wealth as its argument; let $x : [s,\infty) \to [0,\infty)$ be the stochastic timepath of wealth starting at x_s when the consumption function is c;[10] and suppose that $u : (0,\infty) \to \mathbb{R}$ is a utility function which takes consumption as its argument. Then the preferences of self s are given by

$$\begin{aligned} &\mathrm{E}_s\Big[\int_s^\infty D(t-s)\, u(c(x(t)))\, dt\Big] \\ &\quad = \mathrm{E}_s\Big[\underbrace{\int_s^{s+\tau} \delta^{t-s}\, u(c(x(t)))\, dt}_{\text{present}} + \underbrace{\beta \int_{s+\tau}^\infty \delta^{t-s}\, u(c(x(t)))\, dt}_{\text{future}}\Big]. \end{aligned}$$

We therefore define the *continuation*-value function $v : [0,\infty) \to \mathbb{R}$ of self s by the formula

$$v(x_{s+\tau}) = \mathrm{E}_{s+\tau}\Big[\int_{s+\tau}^\infty \delta^{t-s-\tau}\, u(c(x(t)))\, dt\Big];$$

[10]The noise in the asset returns ensures that the dynamics are *uniquely* soluble from *all* initial wealth levels for a *very* wide class of consumption functions. Indeed, suppose that we begin with a Borel measurable function $c : (0,\infty) \to (0,\infty)$ that is locally integrable in $(0,\infty)$. Then, for any $x_0 \in (0,\infty)$, the dynamics are uniquely soluble up to the first time that x hits 0. The only question is therefore what happens when $x_0 = 0$. In order to answer this question, let G be the first hitting time of 0. (It is entirely possible that $G = \infty$, in which case x need not go anywhere near 0.) Then there are two mutually exclusive and exhaustive cases. In the first case, $\liminf_{t\to G-} x(t) = \infty$ with probability one for all $x_0 \in (0,\infty)$. In this case we are free to pick any $c(0) \in (0, y]$. For $c(0) \in (0, y)$, the dynamics will be uniquely soluble starting at 0. This solution will have the property that $x(t) > 0$ for all $t > 0$, and it will be independent of the exact choice of $c(0)$. For $c(0) = y$, the dynamics will have a continuum of solutions. At one extreme, $x(t) > 0$ for all $t > 0$. In this case, the solution coincides with that obtained when $c(0) \in (0, y)$. At the other extreme, $x(t) = 0$ for all $t \geq 0$. In between, there will be an exponentially distributed time H such that $x(t) = 0$ for $t \leq H$ and $x(t) > 0$ for $t > H$. In the second case, $\liminf_{t\to G-} x(t) = 0$ with probability one for all $x_0 \in (0,\infty)$. In this case we are compelled to put $c(0) = y$ if we want the dynamics to have a solution at all. The first case can be thought of as the case of certain accumulation. In this case, it makes sense to require that $c(0) < y$. The second case can be thought of as the case of possible decumulation. In this case, it makes sense to require that $c(0) = y$. Indeed, with these conventions, the dynamics are uniquely soluble for all initial wealth levels.

and we define the *current*-value function $w : [0,\infty) \to \mathbb{R}$ of self s by the formula

$$w(x_s) = \mathrm{E}_s\Big[\underbrace{\textstyle\int_s^{s+\tau} \delta^{t-s}\, u(c(x(t)))\, dt}_{\text{present}} + \underbrace{\beta\, \delta^\tau\, v(x(s+\tau))}_{\text{future}}\Big].$$

The continuation-value function v discounts utility flows by the standard exponential discount factor. The current-value function w discounts utility flows up to the stochastic transition time τ by the standard exponential discount factor, and it discounts the continuation-value obtained at τ by the composite discount factor $\beta\,\delta^\tau$. The component β reflects the one-time discounting that arises from the transition from the present to the future. The component δ^τ reflects standard exponential discounting.

3.3. Equilibrium. Using the notation from the previous subsection, we define equilibrium as follows.[11]

Definition 1. *A consumption function c is an equilibrium of the finite-λ model iff:*

1. *For all consumption functions $\widetilde{c}$ and all $x_0 \in [0,\infty)$, we have*

$$w(x_0) \geq \mathrm{E}\Big[\textstyle\int_0^\tau e^{-\gamma t}\, u(\widetilde{c}(\widetilde{x}(t)))\, dt + \beta\, e^{-\gamma\tau}\, v(\widetilde{x}(\tau))\Big],$$

   where $\widetilde{x} : [0,\infty) \to [0,\infty)$ is the stochastic timepath of wealth starting at x_0 when the consumption function is $\widetilde{c}$.

2. *For all $x_0 \in [0,\infty)$, we have $v(x_0) \geq \frac{1}{\gamma}\, u(y)$.*

The first condition in this definition of equilibrium reflects the fact that the current self maintains control of consumption for the duration of the present — i.e., until the next stochastic transition date τ periods in the future. It could be summarized by saying that, if all future players use the consumption function c then, for all initial wealth levels x_0, the consumption function c is itself a best response for the current self. The second condition, which is purely technical, requires that equilibrium continuation-payoff functions be bounded

[11] Our equilibrium concept is essentially perfect equilibrium in stationary Markov strategies. However, we depart from the usual definition in only allowing deviations to stationary Markov strategies. (The standard definition allows deviations to arbitrary non-stationary and history-dependent strategies.) We do this for expositional convenience. (It should be intuitively clear that the set of equilibria is unaffected.)

below by the payoff function associated with the myopic policy "always consume deterministic labor income y". This requirement rules out equilibria supported by policy functions that generate expected utility of $-\infty$. Such infinitely bad policy functions can technically be equilibria since no single self has an incentive to deviate.[12]

3.4. Characterization of Equilibrium. In this section we give a heuristic derivation of the Bellman system for our problem. There are three parts to this Bellman system: an equation for the continuation-value function of the current self, an equation for the current-value function of the current self, and an instantaneous optimality condition determining the consumption chosen by the current self. Our derivation can be made rigorous in the standard way.

We begin with the equation for the continuation-value function v. Suppose that the current state is x. Then $v(x)$ has two components, namely the current payoff $u(c(x))\,dt$ and the expected discounted continuation payoff $\mathrm{E}[e^{-\gamma\,dt}\,v(x+dx)]$. We therefore have

$$v(x) = u(c(x))\,dt + \mathrm{E}\big[e^{-\gamma\,dt}\,v(x+dx)\big].$$

Multiplying through by $e^{\gamma\,dt}$ and subtracting $v(x)$ from both sides, we obtain

$$(e^{\gamma\,dt}-1)\,v(x) = e^{\gamma\,dt}\,u(c(x))\,dt + \mathrm{E}\big[v(x+dx)-v(x)\big].$$

Now

$$e^{\gamma\,dt} = 1 + \gamma\,dt + O(dt^2)$$

and

$$\mathrm{E}\big[v(x+dx)-v(x)\big] = ((\mu\,x + y - c(x))\,v'(x) + \tfrac{1}{2}\,\sigma^2\,x^2\,v''(x))\,dt + O(dt^2)$$

(cf. Itô's Lemma). Hence, dividing through by dt, letting $dt \to 0$ and suppressing the dependence of v and c on x, we obtain

$$\gamma\,v = u(c) + (\mu\,x + y - c)\,v' + \tfrac{1}{2}\,\sigma^2\,x^2\,v''. \tag{2}$$

The term $\gamma\,v$ represents the expected value of instantaneous changes in v arising from exponential discounting at rate γ; the term $u(c)$ is the flow of utility derived from the consumption

[12]It may be possible to replace the second condition by the weaker requirement that there exists $\eta \in (0, y)$ such that, for all $x_0 \in [0,\infty)$, we have $v(x_0) \geq \frac{1}{\gamma}\,u(\eta)$.

c; the term $(\mu\, x + y - c)\, v'$ is the expected value of instantaneous changes in v arising from the deterministic component of the returns process; and the term $\frac{1}{2}\,\sigma^2\, x^2\, v''$ is the expected value of instantaneous changes in v arising from the stochastic component of the returns process.

Next, we derive the equation for the current-value function w. The derivation is analogous to that of the equation for the continuation-value function v. Suppose that the current state is x. Then we can decompose $w(x)$ into a current payoff and an expected discounted continuation payoff. The current payoff is $u(c(x))\, dt$ as before; but the continuation payoff now depends on whether the transition between the present and the future occurs or not. If this transition does not occur, then the continuation value is $w(x+dx)$. If the transition does occur, the continuation value is $\beta\, v(x + dx)$. Since the probabilities of these two outcomes are $e^{-\lambda\, dt}$ and $1 - e^{-\lambda\, dt}$ respectively, and since the transition is independent of the evolution of wealth, we have

$$\begin{aligned} w(x) &= u(c(x))\, dt + e^{-\lambda\, dt}\, \mathrm{E}\big[e^{-\gamma\, dt}\, w(x + dx)\big] \\ &\quad + (1 - e^{-\lambda\, dt})\, \mathrm{E}\big[e^{-\gamma\, dt}\, \beta\, v(x + dx)\big]. \end{aligned}$$

Proceeding as in the derivation of the equation for v,[13] we therefore obtain

$$\gamma\, w = u(c) + (\mu\, x + y - c)\, w' + \tfrac{1}{2}\, \sigma^2\, x^2\, w'' + \lambda\, (\beta\, v - w). \tag{3}$$

The only difference between equation (3) and equation (2) is that equation (3) contains the additional term $\lambda\, (\beta\, v - w)$. This term is the expected value of the instantaneous change in w arising from the stochastic arrival, with hazard rate λ, of a transition between the present, with current value w, and the future, with continuation value $\beta\, v$.

Finally, we derive an equation for the consumption function c. Suppose that the current state is x. Then consumption is chosen by the current self to maximize the sum of her current payoff and her expected discounted continuation payoff. That is,

$$\begin{aligned} c(x) &= \operatorname*{argmax}_{\widetilde{c}} \big\{ u(\widetilde{c})\, dt + e^{-\lambda\, dt}\, \mathrm{E}\big[e^{-\gamma\, dt}\, w(x + d\widetilde{x})\big] \\ &\quad + (1 - e^{-\lambda\, dt})\, \mathrm{E}\big[e^{-\gamma\, dt}\, \beta\, v(x + d\widetilde{x})\big] \big\}, \end{aligned}$$

[13] Specifically: multiply through by $e^{\gamma\, dt}$; subtract $w(x)$ from both sides; note that $\beta\, v(x + dx) - w(x) = \beta\, (v(x + dx) - v(x)) + \beta\, v(x) - w(x)$; expand $e^{\gamma\, dt}$, $e^{-\lambda\, dt}$, $\mathrm{E}[w(x + dx) - w(x)]$ and $\mathrm{E}[v(x + dx) - v(x)]$; rearrange; divide through by dt; let $dt \to 0$; and suppress the dependence of v, w and c on x!

where

$$d\widetilde{x} = (\mu\, x + y - \widetilde{c})\, dt + \sigma\, x\, dz.$$

Proceeding as in the derivation of the equation for w,[14] we therefore obtain

$$c = \underset{\widetilde{c}}{\operatorname{argmax}} \left\{ u(\widetilde{c}) + (\mu\, x + y - \widetilde{c})\, w' + \tfrac{1}{2}\, \sigma^2\, x^2\, w'' + \lambda\, (\beta\, v - w) \right\}.$$

However, in the present case we can simplify further: the objective is unaffected if we subtract off all the terms that do not depend on $\widetilde{c}$. We therefore arrive at

$$c = \underset{\widetilde{c}}{\operatorname{argmax}} \left\{ u(\widetilde{c}) - \widetilde{c}\, w' \right\}.$$

It follows at once from the first-order conditions for this maximization that

$$u'(c) = w' \tag{4}$$

when $x > 0$, since in this case c is unconstrained; and that

$$u'(c) = \max\{w', u'(y)\} \tag{5}$$

when $x = 0$, since in this case $c \leq y$. In other words, when $x > 0$, consumption is chosen so as to equate the marginal utility of consumption to the marginal value of wealth in the hands of the current self (as measured by the current-value function w); and, when $x = 0$, consumption is chosen so as to equate the marginal utility of consumption to the marginal value of wealth in the hands of the current self, or to $u'(y)$, whichever is the higher.

Combining equations (2-5), we arrive at the following definition:

Definition 2. *The Bellman system of the finite-λ model consists of the pair of ordinary differential equations*

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, v'' + (\mu\, x + y - c)\, v' - \gamma\, v + u(c), \tag{6}$$

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, w'' + (\mu\, x + y - c)\, w' - \gamma\, w + u(c) + \lambda\, (\beta\, v - w) \tag{7}$$

[14] The key point to note is that the objective is unaffected by the positive affine transformations involved in the derivation of the equation for w, namely multiplying by $e^{\gamma\, dt}$, subtracting $w(x)$ and dividing by dt.

and the instantaneous optimality condition

$$u'(c) = \left\{ \begin{array}{ll} w' & \textit{if } x > 0 \\ \max\{w', u'(y)\} & \textit{if } x = 0 \end{array} \right\}. \tag{8}$$

3.5. Equilibrium in the Alternative Model. In this section, we define equilibrium in, and describe the Bellman system of, the alternative model in which the *stochastic* discount function D of the finite-λ model is replaced by the *deterministic* discount function $\overline{D}$. The reader who is not interested in the alternative model may wish to jump immediately to the next section.

Suppose that $c : [0, \infty) \to (0, \infty)$ is a consumption function. Then we may define a '*discounted* value function' $Z : [0, \infty)^2 \to \mathbb{R}$ by the formula

$$Z(s, x_s) = \mathrm{E}\Big[\int_s^\infty \overline{D}(t)\, u(c(x(t)))\, dt \Big],$$

where $x : [s, \infty) \to [0, \infty)$ is the timepath of wealth starting at x_s when the consumption function is c.[15] The value function Z disregards utility flows prior to time s, and it discounts utility flows from time s onwards back to time 0 using the discount factor $\overline{D}(s)$. It can be thought of as the value to self 0 of self s having wealth x_s. More generally, given that the model is stationary, it can be thought of as the value to self t of self $t + s$ having wealth x_s.

Arguing along the same lines as in Section 3.4 above, it can be shown that Z satisfies the partial differential equation

$$0 = \overline{D}\, u(c) + \frac{\partial Z}{\partial s} + (\mu\, x + y - c)\, \frac{\partial Z}{\partial x} + \tfrac{1}{2}\, \sigma^2\, x^2\, \frac{\partial^2 Z}{\partial x^2}, \tag{9}$$

where we have suppressed the dependence of Z on s and x, the dependence of $\overline{D}$ on s and the dependence of c on x. Similarly, since c is chosen by the current self, it satisfies the equation

$$u'(c(x)) = \left\{ \begin{array}{ll} \frac{\partial Z}{\partial x}(0, x) & \text{if } x > 0 \\ \max\left\{\frac{\partial Z}{\partial x}(0, x), u'(y)\right\} & \text{if } x = 0 \end{array} \right\}. \tag{10}$$

In other words, when $x > 0$, consumption is chosen so as to equate the marginal utility of consumption to the marginal value of wealth in the hands of self 0; and, when $x = 0$, consumption is chosen so as to equate the marginal utility of consumption to the marginal

[15] As explained in footnote 10 above, the dynamics are soluble for a very wide class of consumption functions.

value of wealth in the hands of self 0, or to $u'(y)$, whichever is the higher.

These considerations motivate the following definitions.

Definition 3. *A consumption function c is an equilibrium of the alternative model iff:*

1. *For all $x \in [0,\infty)$, $u'(c(x)) = \left\{ \begin{array}{ll} \frac{\partial Z}{\partial x}(0,x) & \text{if } x > 0 \\ \max\left\{\frac{\partial Z}{\partial x}(0,x), u'(y)\right\} & \text{if } x = 0 \end{array} \right\}$.*
2. *For all $s, x \in [0,\infty)$, $Z(s,x) \geq \int_s^\infty \overline{D}(t)\, u(y)\, dt$.*

In other words, while the definition of equilibrium in the finite-λ model involves a full optimality condition and a lower bound on the value function, the definition of equilibrium in the alternative model involves an instantaneous optimality condition and a lower bound on the value function.

Definition 4. *The Bellman system of the alternative model consists of the partial differential equation*

$$0 = \overline{D}\, u(c) + \frac{\partial Z}{\partial s} + (\mu\, x + y - c)\, \frac{\partial Z}{\partial x} + \tfrac{1}{2}\, \sigma^2\, x^2\, \frac{\partial^2 Z}{\partial x^2}$$

and the optimality condition

$$u'(c(x)) = \left\{ \begin{array}{ll} \frac{\partial Z}{\partial x}(0,x) & \text{if } x > 0 \\ \max\left\{\frac{\partial Z}{\partial x}(0,x), u'(y)\right\} & \text{if } x = 0 \end{array} \right\}.$$

In other words, while the definition of the Bellman system of the finite-λ model involves a pair of ordinary differential equations and an instantaneous optimality condition, the definition of the Bellman system of the alternative model involves a partial differential equation and an instantaneous optimality condition.

Finally, recall that $\overline{D}(s) = e^{-\lambda\, s}\, e^{-\gamma\, s} + (1 - e^{-\lambda\, s})\, \beta\, e^{-\gamma\, s}$. (We have not used this fact in arriving at Definitions 3 and 4, which are valid for all discount functions.) Using this, it is easy to show that

$$Z(s,x) = e^{-\lambda\, s}\, e^{-\gamma\, s}\, w(x) + (1 - e^{-\lambda\, s})\, \beta\, e^{-\gamma\, s}\, v(x),$$

and hence

$$\frac{\partial Z}{\partial x}(0,x) = w'(x).$$

We can therefore exploit the characterization of equilibrium in the finite-λ model in terms of the Bellman system of the finite-λ model to see that c is an equilibrium of the finite-λ model if and only if it is an equilibrium of the alternative model.[16]

4. The Instantaneous-Gratification Model

The continuous-time consumption model presented in the last subsection has an immediate advantage over its discrete-time analogue: in the continuous-time model equilibrium consumption functions are everywhere continuous. However, the principal pathology of the discrete-time hyperbolic consumption model remains: there may be intervals on which the consumption function is downward sloping.[17]

Fortunately, we need not focus on the general case of the model. The urge for instant gratification suggests that the present — i.e. the interval $[t, t+\tau)$ during which consumption is highly valued — is very short. Since the arrival rate of τ is λ, this is the same as saying that λ is very large. We are therefore led to consider the limiting case $\lambda \to \infty$, which serves as a proxy for the case where τ is small. We refer to the limiting case as the instantaneous-gratification case, or IG case for short.

Suppose that the triple $(v_\lambda, w_\lambda, c_\lambda)$ solves the Bellman system of the finite-λ model. Suppose further that $(v_\lambda, w_\lambda, c_\lambda) \to (v, w, c)$ as $\lambda \to \infty$. Then, letting $\lambda \to \infty$ in equation (6), we obtain

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - c)\,v' - \gamma\,v + u(c). \tag{11}$$

In other words, v is the expected present discounted value obtained when the discount rate is γ, and when consumption is chosen according to the exogenously given consumption function c. Next, dividing equation (7) through by λ and rearranging, we obtain

$$w_\lambda - \beta\,v_\lambda = \tfrac{1}{\lambda}\left(\tfrac{1}{2}\,\sigma^2\,x^2\,w_\lambda'' + (\mu\,x + y - c_\lambda)\,w_\lambda' - \gamma\,w_\lambda + u(c_\lambda)\right).$$

Hence, letting $\lambda \to \infty$,

$$w - \beta\,v = 0. \tag{12}$$

[16] As far as the lower bound on the value functions is concerned, if $v(x) \geq \frac{1}{\gamma}\,u(y)$ then, since the current self can always choose $c = y$, we must have $w(x) \geq \frac{\gamma+\lambda\,\beta}{\gamma\,(\gamma+\lambda)}\,u(y)$. Combining these two inequalities, we obtain $Z(s,x) \geq \int_s^\infty \overline{D}(t)\,u(y)\,dt$. Conversely, if $Z(s,x) \geq \int_s^\infty \overline{D}(t)\,u(y)\,dt$ then, multiplying both sides by $e^{\gamma\,s}$ and letting $s \to \infty$, we obtain $v(x) \geq \frac{1}{\gamma}\,u(y)$.

[17] The jumps that can occur in equilibrium consumption functions of the discrete-time model are always downward. As such, they are simply mathematically extreme versions of downward slopes. The Brownian noise in the continuous-time model eliminates the mathematical pathology of jumps, but fails to eliminate the economic pathology of downward slopes.

This reflects the fact that, as $\lambda \to \infty$, the discount function drops essentially immediately to a fraction β of its initial value, and that the current-value function w is therefore β times the continuation-value function v. Finally, letting $\lambda \to \infty$ in equation (8), we obtain

$$u'(c) = \left\{ \begin{array}{ll} w' & \text{if } x > 0 \\ \max\{w', u'(y)\} & \text{if } x = 0 \end{array} \right\}. \tag{13}$$

In other words, when $x > 0$, consumption is chosen so as to equate the marginal utility of consumption to the marginal value of wealth in the hands of the current self (as measured by the current-value function w); and, when $x = 0$, consumption is chosen so as to equate the marginal utility of consumption to the marginal value of wealth in the hands of the current self, or to $u'(y)$, whichever is the higher.

This derivation motivates the following definition:

Definition 5. *The Bellman system of the IG model consists of the ordinary differential equation*

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - c)\,v' - \gamma\,v + u(c), \tag{14}$$

and the instantaneous optimality condition

$$u'(c) = \left\{ \begin{array}{ll} \beta\,v' & \textit{if } x > 0 \\ \max\{\beta\,v', u'(y)\} & \textit{if } x = 0 \end{array} \right\}. \tag{15}$$

Equation (14) is identical to equation (11). Equation (15) is obtained by substituting for w in terms of v using equation (12), (i.e. by replacing w with $\beta\,v$). Notice that, in the special case in which $\beta = 1$, the Bellman system of the IG model is precisely the Bellman system of an exponential consumer with utility function u and discount rate γ.

5. Bellman-Equation Equivalence

In the present section, we show that there exists a new utility function $\widehat{u}$ such that the Bellman equation of the IG consumer with the original utility function u is identical to the Bellman equation of the exponential consumer with utility function $\widehat{u}$. We refer to this new consumer as the $\widehat{u}$-consumer. Bellman-equation equivalence (between the IG consumer and the $\widehat{u}$-consumer) is the key argument in our existence and uniqueness proofs for the IG equilibrium (see Section 6).

The current section sets out the technical assumptions of the paper, develops many of the important technical lemmas, and includes the discussion of convex duality that motivates our construction of the $\widehat{u}$ utility function. Some readers may nonetheless wish to skip to Section 6, since the important existence, uniqueness and characterization theorems that can be established using the background machinery developed in the current section can all be found in that section and in Section 7. Doing so will not result in any loss of continuity.

5.1. Assumptions. We shall need the following assumptions:

A1 $u : (0, \infty) \to \mathbb{R}$ is three times continuously differentiable;

A2 $u'(c) > 0$ for all $c > 0$;

A3 there exist $0 < \underline{\rho} \le \overline{\rho} < \infty$ such that $\underline{\rho} \le \frac{-c\,u''(c)}{u'(c)} \le \overline{\rho}$ for all $c > 0$;

A4 there exist $-\infty < \underline{\pi} \le \overline{\pi} < \infty$ such that $\underline{\pi} \le \frac{-c\,u'''(c)}{u''(c)} \le \overline{\pi}$ for all $c > 0$;

A5 $\beta + \underline{\rho} - 1 > 0$;

A6 $(2 - \beta)\,\underline{\rho} - (1 - \beta)\,\overline{\pi} > 0$;

A7 $\gamma > \max_{\rho \in [\underline{\rho}_{\widehat{u}}, \overline{\rho}_{\widehat{u}}]} (1 - \rho)(\mu - \frac{1}{2}\,\rho\,\sigma^2)$, where

$$\underline{\rho}_{\widehat{u}} = \frac{(\beta + \overline{\rho} - 1)\,\overline{\rho}}{(2 - \beta)\,\underline{\rho} - (1 - \beta)\,\overline{\pi}} \text{ and } \overline{\rho}_{\widehat{u}} = \frac{(\beta + \underline{\rho} - 1)\,\underline{\rho}}{(2 - \beta)\,\overline{\rho} - (1 - \beta)\,\underline{\pi}}.$$

Assumption A1 is needed for technical reasons. Assumption A2 means that marginal utility is strictly positive. Assumption A3 means that the relative risk aversion of the consumer may vary with consumption, but is globally bounded. This can be expressed by saying that the consumer has bounded relative risk aversion, or BRRA for short. Analogously, assumption A4 means that the consumer has bounded relative prudence (cf. Kimball 1990), or BRP for short. Assumptions A5 and A6 ensure that the dynamic inconsistency of the IG consumer (as measured by $1 - \beta$) is not too great relative to the bounds on ρ and π.[18] Taken together, they ensure that we can construct a utility function $\widehat{u}$ with the necessary properties. When they are not satisfied, it may happen that the equilibrium consumption rate is infinite. In

[18]It is easy to see that Assumption A5 can be written in the form $1 - \beta < \underline{\rho}$. Moreover it can be shown that, if $\overline{\pi} < \underline{\rho}$, then Assumption A6 can be written in the form $1 - \beta > -\frac{\underline{\rho}}{\underline{\rho} - \overline{\pi}}$; and, if $\overline{\pi} > \underline{\rho}$, then Assumption A6 can be written in the form $1 - \beta < \frac{\underline{\rho}}{\overline{\pi} - \underline{\rho}}$.

other words, the current self may consume all the wealth in a single lump-sum binge, forcing future selves to the subsistence level of consumption.[19] Assumption A7 ensures that the discount rate γ exceeds the rate of growth of the utility of wealth when wealth grows at the risk-adjusted rate of return $\mu - \frac{1}{2}\rho\sigma^2$. It thereby guarantees that expected utility is well defined.[20]

Assumptions A1-A7 simplify dramatically if the consumer has constant relative risk aversion ρ. In that case we have $\underline{\rho} = \overline{\rho} = \rho$, $\underline{\pi} = \overline{\pi} = \rho + 1$ and $\underline{\rho}_{\widehat{u}} = \overline{\rho}_{\widehat{u}} = \rho$. Hence Assumptions A1-A7 reduce to:

B1 $\rho > 0$;

B2 $\beta + \rho - 1 > 0$;

B3 $\gamma > (1-\rho)(\mu - \frac{1}{2}\rho\sigma^2)$.

Assumption B1 means that the utility function is strictly concave. Assumption B2 ensures that the dynamic inconsistency of the IG consumer (as measured by $1-\beta$) is not too great relative to the relevant parameter of the utility function (namely ρ). This assumption would be satisfied in a standard calibration: empirical estimates of the coefficient of relative risk aversion ρ typically lie between $\frac{1}{2}$ and 5; and the short-run discount factor β is typically thought to lie between $\frac{1}{2}$ and 1.[21] However, for completeness, we discuss the case $\beta+\rho-1 < 0$ in Section 8.2. Assumption B3 ensures that the discount rate γ exceeds the rate of growth of the utility of wealth when wealth grows at the risk-adjusted rate of return $\mu - \frac{1}{2}\rho\sigma^2$.

5.2. The Bellman Equation of the IG Consumer. The Bellman equation of the IG consumer is simply the equation obtained by eliminating c from the Bellman system of the IG consumer. In order to derive this equation, let the functions $f_+ : (0,\infty) \to (0,\infty)$ and

[19] A sufficient condition for this to occur is that Assumptions A5 and A6 are strongly reversed, in the sense that $\beta + \overline{\rho} - 1 < 0$ and $(2-\beta)\overline{\rho} - (1-\beta)\underline{\pi} < 0$. See Section 8.2 below.

[20] There are really only two cases involved here. In the first case, $\rho < 1$. In this case, the utility function is unbounded above. Expected utility could therefore be positively infinite if the risk-adjusted rate of return is large and positive. Requiring that γ is large enough compensates for the potentially rapid increase in utility. In the second case, $\rho > 1$. In this case, the utility function is unbounded below. Expected utility could therefore be negatively infinite if the risk-adjusted rate of return is large and negative. Requiring that γ is large enough compensates for the potentially rapid decrease in utility. In the presence of labor income y, this second case should not arise. So Assumption A7 can probably be dispensed with in this case.

[21] See Laibson et al (1998) and Ainslie (1992).

$f_0 : (-\infty, \infty) \to (0, y]$ be defined implicitly by the equations

$$\begin{aligned} u'(f_+(\phi)) &= \phi, \\ u'(f_0(\phi)) &= \max\{\phi, u'(y)\}. \end{aligned}$$

In other words, let $f_+(\phi)$ be the consumption chosen by an exponential consumer who is not liquidity constrained ($x > 0$) and who has marginal value of wealth ϕ; and let be $f_0(\phi)$ be the consumption chosen by an exponential consumer who is liquidity constrained ($x = 0$) and who has marginal value of wealth ϕ. Then $f_+(\beta\,\phi)$ is the consumption chosen by an IG consumer who is not liquidity constrained and who has marginal continuation-value of wealth ϕ; and $f_0(\beta\,\phi)$ is the consumption chosen by an IG consumer who is liquidity constrained and who has marginal continuation-value of wealth ϕ.

Furthermore, let the functions $h_+ : (0, \infty) \to \mathbb{R}$ and $h_0 : (-\infty, \infty) \to \mathbb{R}$ be defined by the formulae

$$\begin{aligned} h_+(\phi) &= u(f_+(\beta\,\phi)) - \phi\, f_+(\beta\,\phi), \\ h_0(\phi) &= u(f_0(\beta\,\phi)) - \phi\, f_0(\beta\,\phi). \end{aligned}$$

In other words, let $h_+(\phi)$ be the flow utility of consumption $u(f_+(\beta\,\phi))$, less the flow cost of spending down wealth $f_+(\beta\,\phi)\,\phi$, for an agent who is not liquidity constrained. Similarly, let $h_0(\phi)$ be the flow utility of consumption $u(f_0(\beta\,\phi))$, less the flow cost of spending down wealth $f_0(\beta\,\phi)\,\phi$, for an agent who is liquidity constrained. Notice that, in both cases, the flow cost of spending down wealth is evaluated using the marginal continuation-value of wealth, not the marginal current-value of wealth: we are evaluating the impact of the consumption decisions of future selves on the current self.

Then, using f_+ to eliminate c from equations (14-15), and taking advantage of the notation h_+, we obtain

$$\begin{aligned} 0 &= \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y - c)\,v' - \gamma\,v + u(c) \\ &= \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y)\,v' - \gamma\,v + u(c) - c\,v' \\ &= \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y)\,v' - \gamma\,v + u(f_+(\beta\,v')) - f_+(\beta\,v')\,v' \\ &= \tfrac{1}{2}\,\sigma^2\,x^2\,v'' + (\mu\,x + y)\,v' - \gamma\,v + h_+(v'). \end{aligned} \tag{16}$$

Similarly, using f_0 to eliminate c from equation (14-15), and taking advantage of the notation

h_0, we obtain

$$0 = y\, v' - \gamma\, v + h_0(v'). \tag{17}$$

Equation (16), with boundary condition (17), is the Bellman equation of the IG consumer.

5.3. The Idea Behind Value-Function Equivalence. Consider a second consumer who is identical to the IG consumer except that she has: (i) utility function $\widetilde{u}$ instead of u; and (ii) present bias $1 \,/\, \widetilde{\beta} = 1$ instead of $1 \,/\, \beta$ (in other words, she is dynamically consistent). Call this consumer the $\widetilde{u}$-consumer.

Let $\widetilde{h}_+$ and $\widetilde{h}_0$ be the analogues, for the $\widetilde{u}$-consumer, of the functions h_+ and h_0. Then, proceeding exactly as above, we can show that her value function $\widetilde{v}$ solves the equation

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, \widetilde{v}'' + (\mu\, x + y)\, \widetilde{v}' - \gamma\, \widetilde{v} + \widetilde{h}_+(\widetilde{v}') \tag{18}$$

with boundary condition (at $x = 0$)

$$0 = y\, \widetilde{v}' - \gamma\, \widetilde{v} + \widetilde{h}_0(\widetilde{v}'). \tag{19}$$

Comparing equations (18-19) with equations (16-17), we see that the only difference between the Bellman equation of the $\widetilde{u}$-consumer and the Bellman equation of the IG consumer is that the former involves the functions $\widetilde{h}_+$ and $\widetilde{h}_0$, whereas the latter involves the functions h_+ and h_0.

What we would like to do, then, is to choose $\widetilde{u}$ in such a way that $\widetilde{h}_+ = h_+$ and $\widetilde{h}_0 = h_0$. For then the Bellman equations of the two consumers are identical, and hence their solutions $\widetilde{v}$ and v coincide. Unfortunately, this is not possible: we can choose $\widetilde{u}$ in such a way that $\widetilde{h}_+ = h_+$, or we can choose $\widetilde{u}$ in such a way that $\widetilde{h}_0 = h_0$, but we cannot choose $\widetilde{u}$ in such a way that both $\widetilde{h}_+ = h_+$ and $\widetilde{h}_0 = h_0$. Fortunately, we can get around this problem. Let $\widehat{u}_+$ be the choice of $\widetilde{u}$ for which $\widetilde{h}_+ = h_+$; let $\widehat{u}_0$ be the choice of $\widetilde{u}$ for which $\widetilde{h}_0 = h_0$; and let $\widehat{u}$ be the *wealth-dependent* utility function that coincides with $\widehat{u}_+$ when $x > 0$ and with $\widehat{u}_0$ when $x = 0$. Then — as we shall explain in more detail in the following sections — the Bellman equation of the $\widehat{u}$-consumer is identical to the Bellman equation of the IG consumer.

5.4. The Bellman Equation of the $\widehat{u}$-Consumer. In order to make the idea of Section 5.3 precise, begin from a pair of utility functions $\widehat{u}_+ : (0, \infty) \to \mathbb{R}$ and $\widehat{u}_0 : (0, y] \to \mathbb{R}$. Define the wealth-dependent utility function $\widehat{u}$ by the formula

$$\widehat{u}(x,c) \equiv \left\{ \begin{array}{ll} \widehat{u}_+(c) & \text{if } x>0 \text{ and } c\in(0,\infty) \\ \widehat{u}_0(c) & \text{if } x=0 \text{ and } c\in(0,y] \end{array} \right\}.$$

Consider the exponential consumer with utility function $\widehat{u}$ and discount rate γ. Call this consumer the $\widehat{u}$-consumer.

Arguing as in Section 3.4 above, it is easy to see that the value function $\widehat{v}$ of the $\widehat{u}$-consumer satisfies the equation

$$0 = \max_{c\in(0,\infty)} \tfrac{1}{2}\,\sigma^2\,x^2\,\widehat{v}'' + (\mu\,x + y - c)\,\widehat{v}' - \gamma\,\widehat{v} + \widehat{u}_+(c) \tag{20}$$

with boundary condition (at $x=0$)

$$0 = \max_{c\in(0,y]} (y-c)\,\widehat{v}' - \gamma\,\widehat{v} + \widehat{u}_0(c). \tag{21}$$

Define the functions $\widehat{h}_+ : (0,\infty)\to\mathbb{R}$ and $\widehat{h}_0 : (-\infty,\infty)\to\mathbb{R}$ by the formulae

$$\begin{aligned} \widehat{h}_+(\phi) &= \max_{c\in(0,\infty)} \widehat{u}_+(c) - \phi\,c, \\ \widehat{h}_0(\phi) &= \max_{c\in(0,y]} \widehat{u}_0(c) - \phi\,c. \end{aligned}$$

Then we see that equation (20) with boundary condition (21) can be written in the form

$$0 = \tfrac{1}{2}\,\sigma^2\,x^2\,\widehat{v}'' + (\mu\,x + y)\,\widehat{v}' - \gamma\,\widehat{v} + \widehat{h}_+(\widehat{v}') \tag{22}$$

with boundary condition (at $x=0$)

$$0 = y\,\widehat{v}' - \gamma\,\widehat{v} + \widehat{h}_0(\widehat{v}'). \tag{23}$$

Equation (22) with boundary condition (23) is the Bellman equation of the $\widehat{u}$-consumer. Comparing equations (22-23) with equations (16-17), we can see that the only difference between the Bellman equation of the $\widehat{u}$-consumer and the Bellman equation of the IG consumer is that the former involves the functions $\widehat{h}_+$ and $\widehat{h}_0$, whereas the latter involves the functions h_+ and h_0.[22]

[22]The difference between the Bellman equation of the $\widetilde{u}$-consumer and the Bellman equation of the $\widehat{u}$-

5.5. Choosing $\widehat{u}_+$. We need to choose $\widehat{u}_+$ in such a way that $\widehat{h}_+ = h_+$. To this end, recall that $\widehat{h}_+$ is defined by the formula

$$\widehat{h}_+(\phi) = \max_{c\in(0,\infty)} \widehat{u}_+(c) - \phi\, c.$$

In other words, $\widehat{h}_+$ is the dual — in the sense of convex analysis — of $\widehat{u}_+$. In order to ensure that $\widehat{h}_+ = h_+$, it therefore suffices to let $\widehat{u}_+$ be the dual of h_+. In other words, it suffices to define $\widehat{u}_+ : (0,\infty) \to \mathbb{R}$ by the formula

$$\widehat{u}_+(c) = \min_{\phi\in(0,\infty)} h_+(\phi) + c\,\phi. \tag{24}$$

For then $\widehat{h}_+$ is the double dual of h_+, and therefore equal to h_+.

In order to verify that this approach works, we need three lemmas.

Lemma 6. *We have:*

1. $h_+ : (0,\infty) \to \mathbb{R}$ *is twice continuously differentiable;*
2. $h'_+(\phi) < 0$ *for all* $\phi \in (0,\infty)$*;*
3. $\overline{\rho}_{\widehat{u}}^{-1} \le \frac{-\phi\, h''_+(\phi)}{h'_+(\phi)} \le \underline{\rho}_{\widehat{u}}^{-1}$ *for all* $\phi \in (0,\infty)$.

In particular: h_+ is strictly decreasing and strictly convex; $h'_+(\phi) \to -\infty$ as $\phi \to 0+$; $h'_+(\phi) \to 0$ as $\phi \to \infty-$; and h_+ is BRRA.

Proof. See Appendix A.1. ■

Remark 7. *It is Assumption A5 which ensures that $h'_+ < 0$, and Assumption A6 which ensures that $h''_+ > 0$.*

Lemma 8. *We have:*

1. $\widehat{u}_+ : (0,\infty) \to \mathbb{R}$ *is twice continuously differentiable;*
2. $\widehat{u}'_+(c) > 0$ *for all* $c \in (0,\infty)$*;*
3. $\underline{\rho}_{\widehat{u}} \le \frac{-c\,\widehat{u}''_+(c)}{\widehat{u}'_+(c)} \le \overline{\rho}_{\widehat{u}}$ *for all* $c \in (0,\infty)$.

---

consumer is that $\widetilde{h}_+$ and $\widetilde{h}_0$ are derived from the same wealth-independent utility function $\widetilde{u}$, whereas $\widehat{h}_+$ and $\widehat{h}_0$ are derived from two different wealth-independent utility functions $\widehat{u}_+$ and $\widehat{u}_0$.

In particular: the dual $\widehat{u}_+$ of h_+ is strictly increasing and strictly concave; $\widehat{u}'_+(c) \to \infty$ as $c \to 0+$; $\widehat{u}'_+(c) \to 0$ as $c \to \infty-$; and $\widehat{u}_+$ is BRRA.

Proof. See Appendix A.2. ■

Remark 9. *Because the slope of h_+ lies in the interval $(-\infty, 0)$, $\widehat{u}_+(c)$ is well defined if and only if $c \in (0, \infty)$.*

Finally, we have:

Lemma 10. *Suppose that $\widehat{u}_+$ is defined by the formula (24). Then $\widehat{h}_+ = h_+$.*

Proof. As implied in the text, this is an instance of Fenchel's convex-duality Theorem. (Cf Rockafellar 1970, Section 31.) ■

5.6. Choosing $\widehat{u}_0$. We also need to choose $\widehat{u}_0$ in such a way that $\widehat{h}_0 = h_0$. We proceed exactly as in Section 5.5. We recall that $\widehat{h}_0$ is defined by the formula

$$\widehat{h}_0(\phi) = \max_{c \in (0,y]} \widehat{u}_0(c) - \phi\, c.$$

In other words, $\widehat{h}_0$ is the dual of $\widehat{u}_0$. We therefore define $\widehat{u}_0 : (0, y] \to \mathbb{R}$ by the formula

$$\widehat{u}_0(c) = \min_{\phi \in (-\infty, \infty)} h_0(\phi) + c\, \phi. \tag{25}$$

In other words, we take $\widehat{u}_0$ to be the dual of h_0. Finally, we note that $\widehat{h}_0$ is then the double dual of h_0, and therefore equal to h_0.

In order to verify that this approach works, we again need three lemmas.

Lemma 11. *We have:*

$$h_0(\phi) = \left\{ \begin{array}{ll} u(y) - \phi\, y & \text{if } \phi \in \big(-\infty, \frac{1}{\beta}\, u'(y)\big] \\ h_+(\phi) & \text{if } \phi \in \big[\frac{1}{\beta}\, u'(y), \infty\big) \end{array} \right\}.$$

Moreover $h_0'\big(\frac{1}{\beta}\, u'(y) -\big) \le h_0'\big(\frac{1}{\beta}\, u'(y) +\big)$.

In other words: h_0 is affine on $\big(-\infty, \frac{1}{\beta}\, u'(y)\big]$ with slope $-y$; h_0 coincides with h_+ on $\big[\frac{1}{\beta}\, u'(y), \infty\big)$; and there is a non-negative jump in the slope of h_0 at $\frac{1}{\beta}\, u'(y)$. In particular, h_0 is strictly decreasing and convex.

Proof. See Appendix A.3. ■

Remark 12. *There is a strictly positive jump in the slope of* h_0 *at* $\frac{1}{\beta}\,u'(y)$ *if and only if* $\beta < 1$.

Lemma 13. *We have*

$$\widehat{u}_0(c) = \left\{ \begin{array}{ll} \widehat{u}_+(c) & \text{if } c \in (0, \psi(y)\,y] \\ \widehat{u}_+(\psi(y)\,y) + (c - \psi(y)\,y)\,\widehat{u}'_+(\psi(y)\,y) & \text{if } c \in (\psi(y)\,y, y] \end{array} \right\},$$

where

$$\psi(y) = \frac{\beta + \rho(y) - 1}{\rho(y)} \in (0, 1)$$

and $\rho(y) = -y\,u''(y)\,/\,u'(y)$. *Moreover* $\widehat{u}_0(y) = u(y)$.

In other words: $\widehat{u}_0$ coincides with $\widehat{u}_+$ on $(0, \psi(y)\,y]$; $\widehat{u}_0$ is affine on $[\psi(y)\,y, y]$ with slope $\widehat{u}'_+(\psi(y)\,y)$; and $\widehat{u}_0$ coincides with u at y. In particular, $\widehat{u}_0$ is strictly increasing and concave.
Proof. See Appendix A.4. ■

Remark 14. *Because the slope of* h_0 *lies in the interval* $[-y, 0)$, $\widehat{u}_0(c)$ *is well defined if and only if* $c \in (0, y]$.

Finally, we have:

Lemma 15. *Suppose that* $\widehat{u}_+$ *is defined by the formula (25). Then* $\widehat{h}_0 = h_0$.

Proof. As implied in the text, this is an instance of Fenchel's convex-duality Theorem. (Cf Rockafellar 1970, Section 31.) ■

Remark 16. *The* $\widehat{u}$*-consumer can be thought of as a consumer who has utility function* $\widehat{u}_+$*, but who receives a utility boost if* $x = 0$ *and* $c \in (\psi(y)\,y, y]$.

6. Existence and Uniqueness

Recall that in Section 5 we constructed a utility function $\widehat{u}$ such that a consumer with this utility function and exponential discount rate γ will have exactly the same Bellman equation as the IG consumer (with utility function u, exponential discount rate γ and present bias $1\,/\,\beta$). This utility function had to be *wealth*-contingent: $\widehat{u} = \widehat{u}_0$ when $x = 0$, and $\widehat{u} = \widehat{u}_+$ when $x > 0$. The following theorem draws together the findings relating the Bellman equation associated with the IG model and the Bellman equation associated with the (dynamically consistent) $\widehat{u}$-consumer.

Theorem 17 [Bellman-Equation Equivalence]**.** *The Bellman equation of the IG consumer is identical to the Bellman equation of the $\widehat{u}$-consumer.*

Proof. Let the functions $h_+ : (0, \infty) \to (0, \infty)$ and $h_0 : (-\infty, \infty) \to (0, \infty)$ be defined as in Section 5.2. Let the functions $\widehat{u}_+ : (0, \infty) \to \mathbb{R}$ and $\widehat{u}_0 : (0, y] \to \mathbb{R}$ be the convex duals of h_+ and h_0, as in Sections 5.5 and 5.6. Let the functions $\widehat{h}_+ : (0, \infty) \to (0, \infty)$ and $\widehat{h}_0 : (-\infty, \infty) \to (0, \infty)$ be the convex duals of $\widehat{u}_+$ and $\widehat{u}_0$, as in Section 5.4. Then $\widehat{h}_+ = h_+$ and $\widehat{h}_0 = h_0$, by Lemmas 10 and 15. In particular, the Bellman equation of the IG consumer is identical to the Bellman equation of the $\widehat{u}$-consumer. ■

Armed with Bellman-equation equivalence, it is easy to prove the key theorem of this section. This theorem exploits the fact that any property of the Bellman equation of the $\widehat{u}$-consumer must also hold for the Bellman equation of the IG consumer, since the two Bellman equations coincide. In particular, many good properties of the dynamically-consistent optimization problem of the $\widehat{u}$-consumer all carry over to the dynamically-inconsistent problem of the IG consumer.

Theorem 18. *The IG model has a unique equilibrium. Moreover the equilibrium value function of the IG consumer coincides with the value function of the $\widehat{u}$-consumer.*

Proof. Since the problem of the $\widehat{u}$-consumer is a standard optimization problem, the Bellman equation of the $\widehat{u}$-consumer has a unique solution $\widehat{v}$. This solution must also be a solution of the Bellman equation of the IG consumer. That is, equilibrium exists in the IG problem. On the other hand, any solution v of the Bellman equation of the IG consumer is also a solution of the Bellman equation of the $\widehat{u}$-consumer. Since $\widehat{v}$ is the unique solution of the Bellman equation of the $\widehat{u}$-consumer, we must have $v = \widehat{v}$. In particular, the equilibrium value function of the IG consumer is unique. ■

This is undoubtedly a powerful result. It means that we can reduce the study of the problem of the IG consumer, which is game-theoretic in nature, to the study of the problem of the $\widehat{u}$-consumer, which is decision-theoretic in nature. However, it must be used with care. Although the two consumers share a common value function, they do not share a common consumption function. In particular, value-function equivalence does not translate into observational equivalence. Instead, consumption of the IG consumer will generally exceed consumption of the $\widehat{u}$-consumer, a point that we develop in Section 7.2.

6.1. The Utility Functions of the Two Consumers. In this section, we characterize the relationship between the utility function u of the IG consumer and the utility function $\widehat{u}$ of the $\widehat{u}$-consumer.

Consider the perspective of a consumer with utility function u who discounts exponentially with discount rate γ and is forced to adopt the (inefficient) equilibrium consumption levels of the IG consumer. Note that the value function v is constructed from exactly this perspective: v represents the integral of exponentially discounted expected utility flows that would be experienced by a consumer with utility function u who was forced to implement the IG consumption rule.

Now consider the perspective of the $\widehat{u}$-consumer. By construction, the value function $\widehat{v}$ represents the exponentially discounted integral of expected utility flows that would be experienced by the $\widehat{u}$-consumer who implements her *optimal* policy.

Suppose (counterfactually) that $\widehat{u}$ were equal to u. Then $\widehat{v} > v$, since $\widehat{v}$ is based on the optimal policy for $\widehat{u}$, while v is based on the suboptimal (IG) policy for u. Of course $\widehat{v} > v$ is a contradiction, since $\widehat{u}$ was constructed so that $\widehat{v} = v$.

This reasoning suggests that $\widehat{u}$ must be less than u. Intuitively, since $\widehat{v} = v$, the utility function $\widehat{u}$ must be less than the utility function u to compensate for the fact that the policy that supports $\widehat{v}$ is chosen optimally while the policy that supports v is suboptimal.

The following two lemmas confirm this intuition.

Lemma 19. $\widehat{u}_+(c) < u(c)$ *for all* $c \in (0, \infty)$.

Intuitively speaking, the IG consumption function exhibits overconsumption whenever $x > 0$. Hence $u(c)$ must be strictly larger than $\widehat{u}_+(c)$ to compensate.

Proof. See Appendix A.5. ■

Lemma 20. $\widehat{u}_0(c) \leq u(c)$ *for all* $c \in (0, y]$, *with equality if and only if* $c = y$.

Intuitively speaking, the IG consumption function exhibits overconsumption when $x = 0$ and $c < y$. Hence $u(c)$ must be strictly larger than $\widehat{u}_0(c)$ when $c < y$.

Proof. See Appendix A.6. ■

6.2. The Case in which u is CRRA. When u has constant relative risk aversion, we can derive a closed-form solution for $\widehat{u}$. Specifically, suppose that u is given by

$$u(c) = \tfrac{1}{1-\rho}\, c^{1-\rho}$$

with $\rho \neq 1$, and that $\beta < 1$.[23] Then it is easy to check that

$$\widehat{u}_+(c) = \tfrac{\psi^\rho}{\beta}\, u(c) \tag{26}$$

$$\widehat{u}_0(c) = \left\{ \begin{array}{ll} \tfrac{\psi^\rho}{\beta}\, u(c) & \text{if } c \in (0, \psi\, y] \\ \tfrac{\psi^\rho}{\beta}\big(u(\psi\, y) + (c - \psi\, y)\, u'(\psi\, y)\big) & \text{if } c \in (\psi\, y, y] \end{array} \right\},$$

where

$$\psi = \frac{\beta + \rho - 1}{\rho}.$$

Moreover $\psi \in (0, 1)$, since $\beta < 1$ and $\beta + \rho - 1 > 0$. In other words, the $\widehat{u}$-consumer can be thought of as an exponential consumer who has utility function $\frac{\psi^\rho}{\beta}\, u$ and discount rate γ, but who receives a utility boost if $x = 0$ and $c \in (\psi\, y, y]$.[24] Figure 4 depicts u, $\widehat{u}_+$ and $\widehat{u}_0$ in the case $\beta = 0.7$ and $\rho = 2$. Among other things, it confirms the relationship among u, $\widehat{u}_+$ and $\widehat{u}_0$ predicted by Lemmas 19 and 20.

7. Some Features of Consumption in the IG Model

In the present section, we investigate the IG model in more detail. We establish the continuity of the consumption function in the interior of the wealth space, we establish a sufficient condition for the monotonicity of the consumption function, and we derive a generalized Euler equation governing the evolution of the marginal utility of consumption. Assumptions A1-A7 will be in force throughout the section.

7.1. Continuity of the Consumption Function. Let v be the value function of the IG consumer, and let c be her consumption function. When $x > 0$, there is non-trivial Wiener noise. Hence v' must be continuous. Moreover u is strictly concave. Hence c is uniquely determined by the first-order condition $u'(c) = \beta\, v'$, and c inherits continuity from v'.

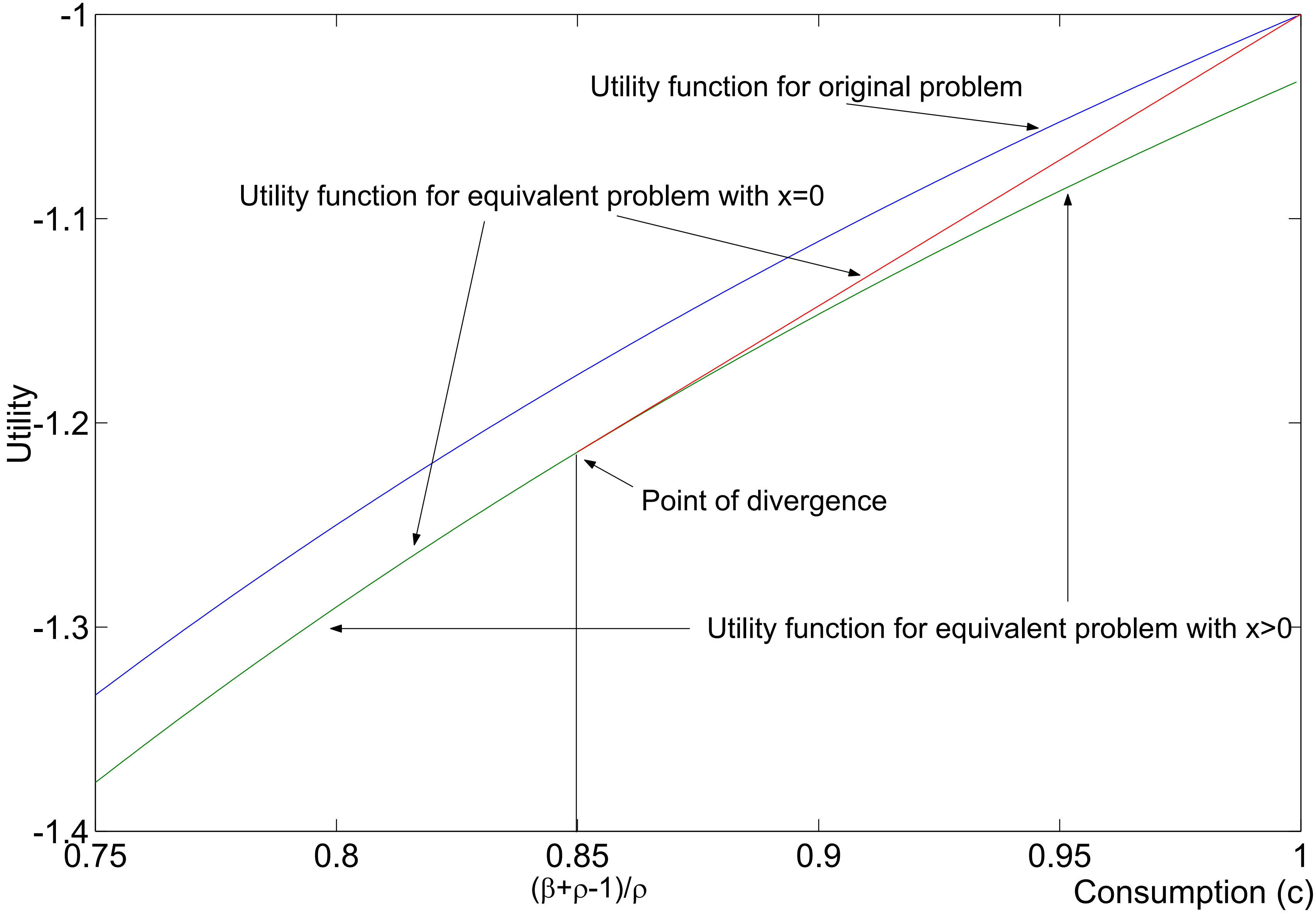
The only remaining question is therefore whether c is continuous at $x = 0$, the point where the liquidity constraint binds.[25] The answer to this question depends on the discount rate γ. There is a critical value γ_1 of γ such that: if $\gamma < \gamma_1$, then $c(0) < y$; and, if $\gamma > \gamma_1$, then $c(0) = y$. In other words: if the IG consumer is patient, then she chooses to save

[23] Ee can also derive a closed-form solution for $\widehat{u}$ in the case of log utility.

[24] The positive scalar $\frac{\psi^\rho}{\beta}$ has no behavioral implications. The $\widehat{u}$-consumer therefore differs from a standard exponential consumer with utility function u and discount rate γ only inasmuch as she receives a utility boost when $x = 0$ and $c \in (\psi\, y, y]$.

[25] Discontinuity in consumption at the moment at which the liquidity constraint starts to bind does not seem empirically counterfactual to us.

Figure 4: Utility functions for equivalent problem (β=.7, ρ=2)



when $x = 0$; and, if she is impatient, then she chooses to consume her labor income forever when $x = 0$. In the case when she is patient, we have $c(0) = c(0+)$. In other words, her consumption behavior at $x = 0$ is simply the limiting case of her consumption behavior when x is small but positive. However, in the case when she is impatient, we have $c(0) < c(0+)$. In other words, when she has a small but positive amount of wealth, she consumes at an unsustainable rate that is sharply curtailed when she runs out of wealth.

More precisely, we have the following theorem that shows that c is always continuous when cash-on-hand is strictly positive, but that c may be discontinuous at the moment the liquidity constraint starts to bind.

Theorem 21. *There exist $\gamma_1 \in (0, \infty)$ and $\bar{c} \in (y, \infty)$ such that:*

1. *if $\gamma < \gamma_1$, then c is continuous on $[0, \infty)$;*
2. *if $\gamma > \gamma_1$, then $c(0) = y$, $c(0+) = \bar{c} > y$ and c is continuous on $(0, \infty)$.*

Moreover $\bar{c}$ is the larger of the two solutions of the equation

$$u'(c) = \beta \, \frac{u(c) - u(y)}{c - y}. \tag{27}$$

Equation (27) can be understood as follows. Let us refer to the moment at which wealth runs out as the 'crunch'. Suppose that the consumption level of the pre-crunch self is c. Then the cost to the pre-crunch self of putting aside an extra dx units of wealth is $u'(c)\,dx$. On the other hand, if the post-crunch self receives a windfall consisting of an extra dx units of wealth, then she can raise her consumption level from y to c for a length of time $dt = dx \,/\, (c - y)$. The benefit to the post-crunch self of this increase in consumption is $(u(c) - u(y))\,dt$, and the benefit to the pre-crunch self is $\beta\,(u(c) - u(y))\,dt$. The pre-crunch self is therefore indifferent between putting aside the extra dx units of wealth and not putting them aside if and only if

$$u'(c)\,dx = \beta\,(u(c) - u(y))\,dt.$$

Substituting for dt and dividing through by dx, we obtain equation (27).

Proof. See Appendix A.7. ■

7.2. Monotonicity of the Consumption Function. The analysis of Section 7.1 shows that c is always non-decreasing at $x = 0$, in the sense that any jump in c at 0 must be non-negative. The main question is therefore whether c is non-decreasing when $x > 0$.

In order to answer this question, let $\widehat{c}$ be the consumption function of the $\widehat{u}$-consumer. Then c and $\widehat{c}$ satisfy the first-order conditions $u'(c) = \beta\, v'$ and $\widehat{u}'_+(\widehat{c}) = v'$. Multiplying the second condition through by β and eliminating $\beta\, v'$, we obtain

$$u'(c) = \beta\, \widehat{u}'_+(\widehat{c}). \tag{28}$$

Since u' and $\widehat{u}'_+$ are both strictly decreasing, it follows that c is non-decreasing if and only if $\widehat{c}$ is non-decreasing.

Now consider the point of view of the $\widehat{u}$-consumer. Intuitively speaking, this consumer will be torn between two options. The first option is to dissave, with a view to bringing forward the time at which she benefits from the utility boost she obtains by consuming only her labor income when her wealth runs out. The second option is to save, with a view to settling into a pattern of steady wealth accumulation. The choice between these two options can be expected to depend on both γ and x. If γ is small, then she can be expected to choose to save for all x; and if γ is large, then she can be expected to choose to dissave for all x. However, for intermediate γ, she can be expected to be torn between the two options. She is more likely to choose to dissave if x small, since she can run down her wealth fairly quickly; and she is more likely to choose to save if x is large, since she can reach the pattern of steady wealth accumulation fairly quickly. Moreover, as x increases and she makes the transition from dissaving to saving, her average propensity to consume can be expected to fall. If this fall is steep enough, it could cause $\widehat{c}$ itself to fall.

Insofar as these considerations relate to the monotonicity of $\widehat{c}$, equation (28) implies that they carry over to c. Specifically, for γ small and γ large, c can be expected to be strictly increasing; and, for intermediate γ and x, c could fall. However, insofar as they relate to whether the $\widehat{u}$-consumer saves or dissaves, they need not carry over. To see why note that, if $\gamma < \gamma_1$, then it follows from Appendix A.1 and part 1 of Theorem 21 that

$$c = \tfrac{1}{\psi(c)}\, \widehat{c} \text{ for all } x \geq 0,$$

where

$$\psi(c) = \tfrac{\beta+\rho(c)-1}{\rho(c)} \in (0,1),$$

$$\rho(c) = \tfrac{-c\, u''(c)}{u'(c)}.$$

Since, $\psi(c)$ is less than unity as long as $\beta < 1$, it follows that Naturally, when u is in the class of utility functions with constant relative risk aversion, then $\rho(c)$ and $\psi(c)$ will not depend on c, so we can express c as an explicit function of $\widehat{c}$

$$c = \frac{1}{\psi}\,\widehat{c} \text{ for all } x \geq 0.$$

Similarly, if $\gamma > \gamma_1$, then it follows from Appendix A.1 and part 2 of Theorem 21 that

$$c(0) = \widehat{c}(0) = y,$$

$$c = \frac{1}{\psi(c)}\,\widehat{c} \text{ for all } x > 0.$$

Overall, then, $c \geq \widehat{c}$ (with strict inequality if $x > 0$). It follows that dissaving by the $\widehat{u}$-consumer translates into dissaving by the IG consumer, but that saving by the $\widehat{u}$-consumer need not translate into saving by the IG consumer.

The following theorem confirms that, if γ is greater than the interest rate μ, then c is indeed strictly increasing. The theorem is presented as a sufficient condition for monotonicity. Once we have derived the deterministic version of the IG model, it will be possible to show that this sufficient condition is also necessary in the deterministic model when γ is near μ. See Section 7.6 below.

Theorem 22. *Suppose that $\gamma \geq \mu$. Then $c' > 0$ when $x > 0$.*

Proof. See Appendix A.8. ■

7.3. Overconsumption when u is CRRA. Let us take as our reference point the exponential consumer who has utility function $\frac{\psi\rho}{\beta}\,u$ and discount rate γ, but who does not receive a utility boost at the origin. Since the factor $\frac{\psi\rho}{\beta}$ has no behavioral significance, this consumer is simply the standard consumer of the standard buffer-stock model. Call her the reference consumer.

Now consider the $\widehat{u}$-consumer. If $\gamma < \gamma_1$, then $\widehat{c}(0) \leq \psi\,y$. In other words, she never takes advantage of the utility boost at $x = 0$. Her consumption function is therefore identical to that of the reference consumer. If $\gamma > \gamma_1$, then $\widehat{c}(0) = y$. In other words, she takes full advantage of the utility boost at $x = 0$. This has two consequences. First, when $x = 0$, her consumption must be at least as high as that of the reference consumer. Second, when $x > 0$, she will increase her consumption above that of the reference consumer, in order to bring forward the time at which she obtains the utility boost. The consumption function of the $\widehat{u}$-consumer is therefore greater than that of the reference consumer, and strictly greater for all $x > 0$.

Finally, consider the IG consumer. If $\gamma < \gamma_1$, then it follows from the discussion of Section 7.2 that

$$c = \tfrac{1}{\psi}\,\widehat{c} \text{ for all } x \geq 0,$$

where

$$\psi = \tfrac{\beta+\rho-1}{\rho} \in (0,1).$$

In other words, c is simply a scalar multiple of $\widehat{c}$. Similarly, if $\gamma > \gamma_1$, then

$$c(0) = \widehat{c}(0) = y,$$

and

$$c = \tfrac{1}{\psi}\,\widehat{c} \text{ for all } x > 0.$$

Overall, then, $c \geq \widehat{c}$ (with strict inequality if $x > 0$).

7.4. The Generalized Euler Equation. Since $u'(c)$ may have a discontinuity at 0, we cannot use Itô's Lemma to study its dynamics. We can, however, use Itô's Lemma to study the dynamics of $m = \beta\, v'$. These dynamics are very closely related to those of $u'(c)$. Indeed, $u'(c) = m$ when $x > 0$. Moreover:

1. if $c(0+) = c(0)$, then the dynamics of m are identical to those of $u'(c)$;
2. if $c(0+) > c(0)$ and $x(0) \in (0, \infty)$, and if T is the first time that x hits 0, then the dynamics of m are identical to those of $u'(c)$ on the interval $(0, T)$; and
3. if $c(0+) > c(0)$ and $x(0) = 0$, then the dynamics of m are identical to those of $u'(c)$.

The two dynamics only differ if $c(0+) > c(0)$ and $x(0) \in (0, \infty)$, in which case $u'(c)$ jumps up at T (whereas m does not jump).

Theorem 23. *We have:*

$$\frac{dm}{m} = \left(\gamma - \mu + \sigma^2\,\rho(c)\,\frac{x\,c'}{c} + (1-\beta)\,c'\right) dt - \sigma\,\rho(c)\,\frac{x\,c'}{c}\,dz$$

if either $x > 0$ *or* $x = 0$ *and* $c(0+) = c(0)$*; and*

$$\frac{dm}{m} = 0$$

if $x = 0$ *and* $c(0+) > c(0)$.

This theorem gives an exact expression for the rate of growth of m. The equation includes deterministic terms (i.e. the terms which include dt) and a stochastic term (i.e. the final term, which includes dz). The stochastic term captures the negative effect that positive wealth shocks have on marginal utility.

The term $\gamma\, dt$ implies that marginal utility rises more quickly the higher the long-run discount rate γ. The term $-\mu\, dt$ implies that marginal utility rises more slowly the higher the rate of return μ. The term $\sigma^2\, \rho(c)\, \frac{x\, c'}{c}\, dt$ captures two separate effects. First, asset income uncertainty σ^2 affects the savings decision. Second, since marginal utility is non-linear in consumption, asset income uncertainty affects the average value of future marginal utility. The net impact of these two effects is always positive. The term $(1 - \beta)\, c'\, dt$ captures the effect of hyperbolic discounting. Naturally, when $\beta = 1$, this effect vanishes and the model coincides with the standard exponential discounting case.

Proof. See Appendix A.9. ■

7.5. The Deterministic IG Model. Up to now we have assumed that the standard deviation of asset returns $\sigma > 0$. In other words, we have been focussing on the *stochastic* IG model. In the present section, we investigate the case $\sigma = 0$. In other words, we focus on the *deterministic* IG model. We show that, by viewing the deterministic IG model as a limiting case of the stochastic IG model, we are able to pinpoint a unique value function for the deterministic IG model. More precisely, we have the following theorem. The proof, which follows standard lines, is omitted.

Theorem 24. *Let* v_σ *be the value function of the stochastic IG model. Then:*

1. *there is a continuous function* $v : [0, \infty) \to \mathbb{R}$ *such that* $v_\sigma \to v$ *uniformly on compact subsets of* $[0, \infty)$ *as* $\sigma \to 0+$*;*
2. v *is the unique viscosity solution*[26] *of the ordinary differential equation*

$$0 = (\mu\, x + y)\, v' - \gamma\, v + \widehat{h}_+(v') \tag{29}$$

[26] See Crandall et al (1992) for a "user's guide" to viscosity solutions.

with boundary condition

$$0 = y\,v' - \gamma\,v + \widehat{h}_0(v') \tag{30}$$

at $x = 0$. ■

Equation (29) with boundary condition (30) is the Bellman equation of the deterministic IG model.

The equilibrium consumption function c of the deterministic IG model can be determined from the value function v using the first-order condition. More precisely, we have

$$u'(c) = \left\{ \begin{array}{ll} \beta\,v' & \text{if } x > 0 \\ \max\{\beta\,v', u'(y)\} & \text{if } x = 0 \end{array} \right\}.$$

In other words, by letting $\sigma \to 0+$ in the stochastic IG model, we select a unique sensible equilibrium of the deterministic IG model.

Remark 25. *Krusell and Smith (2000) consider a deterministic discrete-time hyperbolic consumption model. They show that equilibrium is indeterminate in their model. Our results suggest that this indeterminacy could be resolved by a refinement analogous to the one that we have used here.*

Remark 26. *The Bellman equation of the deterministic IG model is simpler than than that of the stochastic IG model: it is a first-order ordinary differential equation, whereas the latter is a second-order ordinary differential equation.*

7.6. The Deterministic IG Model when u is CRRA. In the present section, we shall investigate the deterministic IG model in the case that u is CRRA. More precisely, we shall make the following assumptions:[27]

C1 $\sigma = 0$;

C2 $u(c)$ takes the form $\frac{1}{1-\rho}\,c^{1-\rho}$ with $\rho \neq 1$;

C3 $\mu > 0$.

[27] Assumptions C1-C3 are made in addition to the standard assumptions A1-A7. The latter accordingly reduce to Assumptions B1-B3.

Under these assumptions, the Bellman equation possesses a symmetry that allows us to transform it from a *non-autonomous* ordinary differential equation into an *autonomous* ordinary differential equation. We are therefore able to provide a complete analysis of equilibrium in this case.

Among other things, this analysis shows that there there are critical values γ_1 and γ_2 of the discount rate γ such that, if $\gamma < \gamma_1$ or $\gamma > \gamma_2$, then c is strictly increasing; and if $\gamma_1 < \gamma < \gamma_2$, then c first increases, then decreases, then increases again. Indeed, it shows that $\gamma_1 = \beta\,\mu$ and $\gamma_2 = \mu$. It also shows that, for $\gamma < \gamma_2$, the IG consumer chooses to dissave. In particular, c exceeds $\widehat{c}$ by enough to turn saving by the $\widehat{u}$-consumer into dissaving by the IG consumer when $\gamma_1 < \gamma < \gamma_2$.

In the deterministic IG model, total income is $\mu\,x+y$. If c is the equilibrium consumption function, we therefore define the average propensity to consume APC of the IG consumer by the formula

$$APC(x) = \frac{c(x)}{\mu\,x + y}.$$

We then have:

Theorem 27. *Suppose that Assumptions C1-C3 hold. Then there exists $\gamma_3 \in (\mu, \infty)$ such that:*

1. *If $\gamma \in ((1-\rho)\,\mu, \beta\,\mu)$, then APC is constant and strictly less than 1. In particular, c is strictly increasing and affine.*

2. *If $\gamma \in (\beta\,\mu, \mu)$, then there exists $x_1 \in (0, \infty)$ such that APC jumps up at 0, is strictly decreasing and strictly greater than 1 on $(0, x_1)$, jumps down at x_1, and is constant and strictly greater than 1 on (x_1, ∞). Moreover c jumps up at 0, is strictly decreasing on $(0, x_1)$, jumps down at x_1, and is strictly increasing and affine on (x_1, ∞).*

3. *If $\gamma \in (\mu, \gamma_3)$, then APC jumps up at 0, and is strictly decreasing and strictly greater than 1 on $(0, \infty)$. Moreover c jumps up at 0, and is strictly increasing on $(0, \infty)$.*

4. *If $\gamma \in (\gamma_3, \infty)$, then APC jumps up at 0, and is strictly increasing and strictly greater than 1 on $(0, \infty)$. Moreover c jumps up at 0, and is strictly increasing on $(0, \infty)$.*

In particular, the condition $\gamma \geq \mu$ used in the proof of monotonicity of the consumption function is necessary.

Proof. See Appendix A.10 ■

Remark 28. *The analysis of this section can easily be extended to the cover the case $\rho = 1$.*

Remark 29. *There is a unique solution to the dynamics even when $\gamma \in (\beta\,\mu, \mu)$. Indeed, in this case we have $APC > 1$, and x therefore decreases strictly with time.*

8. Derivation of the IG Model Revisited

8.1. A Limit Theorem. Our derivation of the IG model from the finite-λ model in Section 4 was deliberately heuristic. The relationship between the two models can, however, be made rigorous. The following theorem, which we state without proof, gives the flavor of the link between the two models.

Theorem 30. *Suppose that Assumptions A1-A7 hold. Then there exists $\lambda_0 \in (0, \infty)$ such that, for all $\lambda \in [\lambda_0, \infty)$, the finite-$\lambda$ model possesses a unique equilibrium c_λ. If w_λ is the current-value function associated with c_λ, then w_λ is continuous on $[0, \infty)$. Moreover $\frac{1}{\beta}\,w_\lambda$ converges uniformly on compact subsets of $[0, \infty)$ as $\lambda \to \infty$ to a limit function v, which is the unique viscosity solution*[28] *of the Bellman equation (16) for $x \in (0, \infty)$, with boundary condition (17) at $x = 0$.* ■

8.2. A Complementary Theorem. Theorem 30 covers the case in which u has constant relative risk aversion $\rho > 1 - \beta$. It is also possible to prove a limit theorem that covers the case in which u has constant relative risk aversion $\rho < 1 - \beta$. In order to formulate such a theorem, we introduce the following assumptions, which complement Assumptions A5 and A6:

A5′ $\beta + \overline{\rho} - 1 < 0$;

A6′ $(2 - \beta)\overline{\rho} - (1 - \beta)\underline{\pi} < 0$.

The theorem, which we state without proof, is then as follows.

Theorem 31. *Suppose that Assumptions A1-A4, A5′, A6′ and A7 hold. Then there exists $\lambda_0 \in (0, \infty)$ such that, for all $\lambda \in [\lambda_0, \infty)$, the finite-$\lambda$ model possesses a unique equilibrium c_λ. If w_λ is the current-value function associated with c_λ, then w_λ is continuous on $[0, \infty)$. Moreover $\frac{1}{\beta}\,w_\lambda \to \frac{1}{\gamma}\,u(y)$ uniformly on compact subsets of $[0, \infty)$ as $\lambda \to \infty$.* ■

[28] See Crandall et al (1992) for a "user's guide" to viscosity solutions.

This theorem reflects the following behavior. For large λ, the consumer quickly consumes all her wealth. Thereafter, she consumes her labor income. Because her utility function has the property that $\frac{1}{c}\,u(c) \to 0$ as $c \to \infty$, the initial consumption binge does not contribute much to the expected present discounted value of her consumption flow. It is the subsequent consumption of her labor income that matters. The expected present discounted value of this consumption is of course simply $\frac{1}{\gamma}\,u(y)$.

This behavior arises because, when $\rho < 1-\beta$, the utility function is not sufficiently bowed to dampen the feedback effects that arise in hyperbolic models. Instead, these feedback effects drive consumption to infinity. In effect, the current self knows that subsequent selves are going to consume at a very high rate, and therefore chooses to consume at a very high rate herself in order to preempt the consumption by the later selves.

8.3. A Stronger Limit Theorem. Finally, note that Theorem 30 continues to hold when Assumptions A5 and A6 are replaced by the following, significantly weaker, assumptions:

A5″ $\beta + \liminf_{c\to\infty} \rho(c) - 1 > 0$;

A6″ $(2-\beta) \liminf_{c\to\infty} \rho(c) - (1-\beta) \limsup_{c\to\infty} \pi(c) > 0$.

These assumptions ensure that $\widehat{h}_+$ is decreasing and convex near 0. This is enough to ensure that consumption remains bounded as $\lambda \to \infty$. These assumptions are, however, consistent with $\widehat{h}_+$ being increasing or concave away from 0. In other words, for some BRRA utility functions, the IG problem is not value-function equivalent to any exponential consumption problem.

9. Conclusions

We have described a continuous-time model of quasi-hyperbolic discounting. Our model allows for a general class of preferences, includes liquidity constraints, and places no restrictions on equilibrium policy functions. The model is also psychologically relevant. We take the phrase 'instantaneous gratification' literally. We analyze a model in which individuals prefer gratification in the present instant discretely more than consumption in the momentarily delayed future. In this simple setting, equilibrium is unique and the consumption function is continuous. When the long-run discount rate weakly exceeds the interest rate, the consumption function is also monotonic. All of the pathologies that characterize discrete-time quasi-hyperbolic models vanish.

10. References

Ainslie, George. 1992 *Picoeconomics.* Cambridge: Cambridge University Press.

Angeletos, George-Marios, David Laibson, Andrea Repetto, Jeremy Tobacman, and Stephen Weinberg (2001) "Hyperbolic discounting, wealth accumulation, and consumption." Forthcoming, *Journal of Economic Perspectives.*

Akerlof, George A. 1991. "Procrastination and Obedience." *American Economic Review. (Papers and Proceedings),* pp. 1-19.

Barro, Robert. 1999. "Laibson Meets Ramsey in the Neoclassical Growth Model." *Quarterly Journal of Economics.* 114(4), pp. 1125-52.

Carroll, Christopher D. 1992. "The Buffer Stock Theory of Saving: Some Macroeconomic Evidence." *Brookings Papers on Economic Activity.* 2:1992, pp. 61-156.

———. 1997. "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis." *Quarterly Journal of Economics.* 112, pp. 1-57.

_____ and Miles Kimball. 1996. "On the Concavity of the Consumption Function," *Econometrica,* 64(4), pp. *981-992*

Crandall, M. G., H. Ishii, P. L. Lions. 1992. "User's Guide to Viscosity Solutions of Second Order Partial Differential Equations," *Bulletin of the American Mathematical Society*, vol 27(1), pp 1-67.

Deaton, Angus. 1991. "Saving and Liquidity Constraints." *Econometrica.* 59, pp. 1221-48.

DellaVigna, Stefano, and Ulrike Malmendier (2004) "Self-Control in the Marketplace," Quarterly Journal of Economics,

Harris, Christopher, and David Laibson. 2001a. "Dynamic Choices of Hyperbolic Consumers." *Econometrica.*

_____ and _____. 2001b. "Hyperbolic Discounting and Consumption." Forthcoming *Proceedings of the 8th World Congress of the Econometric Society.*

Kimball, Miles. 1990. "Precautionary Saving in the Small and in Large," *Econometrica*, 58(1), pp. 53-73.

Krusell, Per and Tony Smith. 2000. "Consumption-Savings Decisions with Quasi-Geometric Discounting," *Econometrica.*

Laibson, David. 1997a. "Golden Eggs and Hyperbolic Discounting." *Quarterly Journal of Economics.* 62: 2, pp. 443-478.

———. 1997b. "Hyperbolic Discount Functions and Time Preference Heterogeneity." Harvard mimeo.

———, Andrea Repetto, and Jeremy Tobacman. 1998. "Self-Control and Saving for Retirement," *Brookings Papers on Economic Activity* 1, pp.91-196.

Loewenstein, George, and Drazen Prelec. 1992. "Anomalies in Intertemporal Choice: Evidence and an Interpretation." *Quarterly Journal of Economics.* 57, pp. 573-98.

Luttmer, Erzo and Thomas Mariotti. 2003. "Subjective Discount Factors," *Journal of Political Economy.*

Morris, Stephen and Andrew Postlewaite. 1997. "Observational Implications of Nonexponential Discounting," mimeo.

O'Donoghue, Ted, and Matthew Rabin. 1999a. "Doing It Now or Later." *American Economic Review,* 89(1), pp. 103-124.

______ and _______ 1999b. "Incentives for Procrastinators," *Quarterly Journal of Economics*, 114(3), pp. 769-816.

Phelps, E. S., and R. A. Pollak. 1968. "On Second-best National Saving and Game-equilibrium Growth." *Review of Economic Studies.* 35, pp. 185-199.

Rockafellar, R. Tyrrell. 1970. *Convex Analysis.* Princeton University Press: Princeton.

Strotz, Robert H. 1956. "Myopia and Inconsistency in Dynamic Utility Maximization." *Review of Economic Studies.* 23, pp. 165-180.

A. Appendix

A.1. Proof of Lemma 6. Let the function $\widetilde{h}_+ : (0,\infty) \to (0,\infty)$ be defined by the formula

$$\widetilde{h}_+(\phi) = u(f_+(\phi)) - \phi\, f_+(\phi).$$

(This notation is consistent with the notation of Section 5.3 if we put $\widetilde{u} = u$.) Then:

$$\begin{aligned}
h_+(\phi) &= u(f_+(\beta\,\phi)) - f_+(\beta\,\phi)\,\phi \\
&= u(f_+(\beta\,\phi)) - \beta\, f_+(\beta\,\phi)\,\phi - (1-\beta)\, f_+(\beta\,\phi)\,\phi \\
&= \widetilde{h}_+(\beta\,\phi) - (1-\beta)\, f_+(\beta\,\phi)\,\phi.
\end{aligned}$$

Hence

$$\begin{aligned}
h_+'(\phi) &= \beta\,\widetilde{h}_+' - (1-\beta)\, f_+ - (1-\beta)\,\beta\, f_+'\,\phi \\
&= -\beta\, f_+ - (1-\beta)\, f_+ - (1-\beta)\,\beta\, f_+'\,\phi \\
&= -f_+ - (1-\beta)\, f_+'\,\beta\,\phi \\
&= -f_+\left(1 + (1-\beta)\,\frac{f_+'\,\beta\,\phi}{f_+}\right) \\
&= -f_+\left(1 + (1-\beta)\,\frac{u'(f_+)}{f_+\,u''(f_+)}\right) \\
&= -f_+\left(1 - \frac{1-\beta}{\rho(f_+)}\right) \\
&= \frac{-(\beta + \rho(f_+) - 1)\, f_+}{\rho(f_+)}
\end{aligned}$$

where we have suppressed the dependence of $\widetilde{h}_+$ and f_+ on $\beta\,\phi$. Assumption A5 therefore implies that $h_+' < 0$.

Second, in the course of the previous paragraph we showed that

$$h_+'(\phi) = -f_+ - (1-\beta)\, f_+'\,\beta\,\phi.$$

Hence

$$\begin{aligned}
h''_+(\phi) &= -\beta\, f'_+ - (1-\beta)\, f'_+\, \beta - (1-\beta)\, \beta\, f''_+\, \beta\, \phi \\
&= -\beta\, f'_+ \left(1 + (1-\beta)\left(1 + \frac{f''_+\, \beta\, \phi}{f'_+}\right)\right) \\
&= \frac{-\beta}{u''(f_+)} \left(1 + (1-\beta)\left(1 - \frac{u'''(f_+)\, u'(f_+)}{u''(f_+)^2}\right)\right) \\
&= \frac{-\beta}{u''(f_+)} \left(1 + (1-\beta)\left(1 - \frac{\pi(f_+)}{\rho(f_+)}\right)\right) \\
&= \frac{-\beta}{u''(f_+)\, \rho(f_+)} \left((2-\beta)\, \rho(f_+) - (1-\beta)\, \pi(f_+)\right).
\end{aligned}$$

Assumption A6 therefore implies that $h''_+(\phi) > 0$.

Third, using the final expressions obtained above for $h'_+(\phi)$ and $h''_+(\phi)$, we have

$$\frac{-\phi\, h''_+(\phi)}{h'_+(\phi)} = \frac{(2-\beta)\, \rho(f_+) - (1-\beta)\, \pi(f_+)}{(\beta + \rho(f_+) - 1)\, \rho(f_+)}.$$

Hence

$$\overline{\rho}_{\widehat{u}}^{-1} \leq \frac{-\phi\, h''_+(\phi)}{h'_+(\phi)} \leq \underline{\rho}_{\widehat{u}}^{-1},$$

as required. ■

A.2. Proof of Lemma 8. Put

$$g_+(c) = \operatorname{argmin}_{\phi \in (0,\infty)} h_+(\phi) + c\, \phi.$$

Then

$$\begin{aligned}
\widehat{u}'_+(c) &= g_+(c), \\
\widehat{u}''_+(c) &= -1 \,/\, h''_+(g_+(c))
\end{aligned}$$

and

$$\frac{-c\, \widehat{u}''_+(c)}{\widehat{u}'_+(c)} = \frac{h'_+(g_+(c))}{-g_+(c)\, h''_+(g_+(c))}.$$

We may therefore apply part 3 of Lemma 6 to conclude that

$$\underline{\rho}_{\widehat{u}} \le \frac{-c\,\widehat{u}''_+(c)}{\widehat{u}'_+(c)} \le \overline{\rho}_{\widehat{u}},$$

as required. ■

A.3. Proof of Lemma 11. The first statement is immediate from the definition of h_0. It implies that

$$\begin{aligned} h'_0(\tfrac{1}{\beta}\,u'(y)+) = h'_+(\tfrac{1}{\beta}\,u'(y)) &= \frac{-(\beta+\rho(f_+(u'(y)))-1)\,f_+(u'(y))}{\rho(f_+(u'(y)))} \\ &= -\left(\frac{\beta+\rho(y)-1}{\rho(y)}\right) y \ge -y = h'_0(\tfrac{1}{\beta}\,u'(y)-). \end{aligned}$$

This completes the proof of the lemma. ■

A.4. Proof of Lemma 13. We have

$$h_0(\phi) = \left\{ \begin{array}{ll} u(y) - \phi\, y & \text{if } \phi \in \big(-\infty, \tfrac{1}{\beta}\,u'(y)\big] \\ h_+(\phi) & \text{if } \phi \in \big[\tfrac{1}{\beta}\,u'(y), \infty\big) \end{array} \right\}$$

and

$$h'_0(\phi) \left\{ \begin{array}{ll} = -y & \text{if } \phi \in \big(-\infty, \tfrac{1}{\beta}\,u'(y)\big) \\ \in [-y, -\psi\, y] & \text{if } \phi = \tfrac{1}{\beta}\,u'(y) \\ = h'_+(\phi) & \text{if } \phi \in \big(\tfrac{1}{\beta}\,u'(y), \infty\big) \end{array} \right\}.$$

Hence

$$\widehat{u}_0(c) = \min_{\phi\in(-\infty,\infty)} h_+(\phi) + c\,\phi = \widehat{u}_+(c)$$

if $c \in (0, \psi\, y)$ and

$$\begin{aligned} \widehat{u}_0(c) &= h_+\big(\tfrac{1}{\beta}\,u'(y)\big) + c\,\tfrac{1}{\beta}\,u'(y) \\ &= h_+\big(\tfrac{1}{\beta}\,u'(y)\big) + \psi\, y\,\tfrac{1}{\beta}\,u'(y) + (c - \psi\, y)\,\tfrac{1}{\beta}\,u'(y) \\ &= \widehat{u}_+(\psi\, y) + (c - \psi\, y)\,\widehat{u}'_+(\psi\, y) \end{aligned}$$

if $c \in [\psi\, y, y]$. Finally,

$$\begin{aligned}\widehat{u}_0(y) &= h_+\big(\tfrac{1}{\beta}\, u'(y)\big) + y\, \tfrac{1}{\beta}\, u'(y)\\ &= u(y) - y\, \tfrac{1}{\beta}\, u'(y) + y\, \tfrac{1}{\beta}\, u'(y)\\ &= u(y).\end{aligned}$$

This completes the proof of the lemma. ■

A.5. Proof of Lemma 19. Let the function $\widetilde{h}_+ : (0,\infty) \to (0,\infty)$ be defined by the formula

$$\widetilde{h}_+(\phi) = u(f_+(\phi)) - \phi\, f_+(\phi).$$

(This notation is consistent with the notation of Section 5.3 if we put $\widetilde{u} = u$.) Then:

$$h_+(\phi) = u(f_+(\beta\,\phi)) - \phi\, f_+(\beta\,\phi) < u(f_+(\phi)) - \phi\, f_+(\phi) = \widetilde{h}_+(\phi)$$

(by definition of h_+, since $f_+(\phi)$ is the unique value of c that maximizes $u(c) - \phi\, c$ and by definition of $\widetilde{h}_+$, respectively); and hence

$$\widehat{u}_+(c) = \min_{\phi\in(0,\infty)} h_+(\phi) + c\,\phi < \min_{\phi\in(0,\infty)} \widetilde{h}_+(\phi) + c\,\phi = u(c)$$

(by definition of $\widehat{u}_+$, because $\widetilde{h}_+ > h_+$ and by convex duality, respectively). ■

A.6. Proof of Lemma 20. Let the function $\widetilde{h}_0 : (-\infty,\infty) \to \mathbb{R}$ be defined by the formula

$$\widetilde{h}_0(\phi) = u(f_0(\phi)) - \phi\, f_0(\phi).$$

(This notation is consistent with the notation of Section 5.3 if we put $\widetilde{u} = u$.) Then

$$h_0(\phi) = u(f_0(\beta\,\phi)) - \phi\, f_0(\beta\,\phi) \le u(f_0(\phi)) - \phi\, f_0(\phi) = \widetilde{h}_0(\phi),$$

with equality if and only if $\phi \in (-\infty, u'(y))$. Hence

$$\widehat{u}_0(c) = \min_{\phi\in(-\infty,\infty)} h_0(\phi) + c\,\phi \le \min_{\phi\in(-\infty,\infty)} \widetilde{h}_0(\phi) + c\,\phi = u(c).$$

Now, suppose that $c \in (0, y)$. Then $u(c)$ is attained for a ϕ_* for which $\widetilde{h}_0(\phi_*) > h_0(\phi_*)$. Hence

$$u(c) = \widetilde{h}_0(\phi_*) + c\,\phi_* > h_0(\phi_*) + c\,\phi_* \geq \widehat{u}_0(c).$$

On the other hand, if $c = y$ then $\widehat{u}_0(c)$ is attained for a ϕ_* for which $\widetilde{h}_0(\phi_*) > h_0(\phi_*)$. (Specifically, $\widehat{u}_0(c)$ is attained for any $\phi_* \leq u'(y)$.) A fortiori, $u(c)$ is also attained for any such ϕ_*. Hence

$$u(c) = \widetilde{h}_0(\phi_*) + c\,\phi_* = h_0(\phi_*) + c\,\phi_* = \widehat{u}_0(c).$$

Overall, then, $\widehat{u}_0(c) = u(c)$ if and only if $c = y$. ■

A.7. Proof of Theorem 21. Put $\phi = v'(0)$. Then (since $v'(0)$ is by definition the limit of $v'(x)$ as $x \to 0$) we must have both

$$0 = y\,\phi - \gamma\,v(0) + h_+(\phi)$$

(from equation (16)) and

$$0 = y\,\phi - \gamma\,v(0) + h_0(\phi).$$

(from equation (17)). Eliminating $\gamma\,v(0)$ between these two equations, we obtain

$$y\,\phi + h_0(\phi) = y\,\phi + h_+(\phi).$$

This equation has a single isolated solution $\phi = \overline{\phi} \in \left(0, \frac{1}{\beta}\,u'(y)\right)$, and a continuum of solutions $\phi \in \left[\frac{1}{\beta}\,u'(y), \infty\right)$. The isolated solution corresponds to the case in which the consumer is impatient, and chooses to consume her labor income at $x = 0$. The continuum of solutions correspond to the case in which the consumer is patient, and chooses to save at $x = 0$. At the isolated solution, we have

$$\gamma\,v(0) = y\,\phi + h_0(\phi) = y\,\phi + u(y) - \phi\,y = u(y)$$

(cf. Lemma 11). At a solution in the continuum, we have

$$\gamma\,v(0) = y\,\phi + h_+(\phi).$$

In particular, $\gamma\,v(0)$ is strictly increasing in ϕ.

Next, consider the point of view of the $\widehat{u}$-consumer. She is a standard optimizer. It is therefore obvious that the flow equivalent $\gamma\, v(0)$ of the value to her of having $x = 0$ is non-decreasing in γ. Moreover, there must be a critical value γ_1 of her discount rate γ such that: $\gamma\, v(0)$ is strictly decreasing in γ for $\gamma < \gamma_1$; and $\gamma\, v(0)$ is constant and equal to $u(y)$ when $\gamma > \gamma_1$. In other words, for $\gamma < \gamma_1$ we will obtain the appropriate solution $\phi \in \left(\frac{1}{\beta}\, u'(y), \infty\right)$, and for $\gamma > \gamma_1$ we will obtain the isolated solution $\phi = \overline{\phi}$.

Finally, put $\overline{c} = f_+(\beta\, \overline{\phi})$ and consider the point of view of the IG consumer. If $\gamma < \gamma_1$, then we have

$$c(0) = f_0(\beta\, \phi) = f_+(\beta\, \phi) = c(0+).$$

If $\gamma > \gamma_1$, then we have

$$\begin{aligned} c(0) &= f_0(\beta\, \phi) = y \\ c(0+) &= f_+(\beta\, \phi) = \overline{c}. \end{aligned}$$

Now,

$$y\, \phi + h_0(\phi) = \left\{ \begin{array}{ll} u(y) & \text{if } \phi \in \left(0, \frac{1}{\beta}\, u'(y)\right) \\ y\, \phi + h_+(\phi) & \text{if } \phi \in \left[\frac{1}{\beta}\, u'(y), \infty\right) \end{array} \right\}.$$

Hence $\overline{\phi}$ is the smaller of the two solutions of the equation

$$u(y) = y\, \phi + h_+(\phi).$$

Noting that $h_+(\phi) = u(f_+(\beta\, \phi)) - \phi\, f_+(\beta\, \phi)$, making the substitution $\phi = \frac{1}{\beta}\, u'(c)$ and rearranging, this equation can be written in the form (27). Since u' is strictly decreasing, $\overline{c}$ is the larger of the two solutions of this equation. ■

A.8. Proof of Theorem 22. As explained in the text, it suffices to show that $\widehat{c}' > 0$ when $x > 0$. As a first step, note that $\widehat{c}$ satisfies the first-order condition $\widehat{u}'_+(\widehat{c}) = v'$ when $x > 0$. Hence $\widehat{u}''_+(\widehat{c})\, \widehat{c}' = v''$. Hence $\widehat{c}' > 0$ if and only if $v'' < 0$. Next, differentiating equation (22) with respect to x, we obtain

$$0 = \tfrac{1}{2}\, \sigma^2\, x^2\, v''' + (\mu\, x + y)\, v'' - \gamma\, v' + \sigma^2\, x\, v'' + \mu\, v' + \widehat{h}'_+(v')\, v''$$

or

$$v''' = \frac{2}{\sigma^2\, x^2}\, ((\gamma - \mu)\, v' - ((\mu + \sigma^2)\, x + y + \widehat{h}'_+(v'))\, v'').$$

In particular, if $v'' = 0$, then

$$v''' = \frac{2}{\sigma^2 x^2} (\gamma - \mu) v' \geq 0.$$

Hence, if there exists $x_1 \in (0, \infty)$ such that $v''(x_1) \geq 0$, then $v'' \geq 0$ on (x_1, ∞). In other words, v grows at least linearly. But this is impossible. Under our standing assumptions, there exists $k_1 > 0$ such that $\widehat{u}_0$ and $\widehat{u}_+$ are dominated by the CRRA utility function $\overline{u} : (0, \infty) \to \mathbb{R}$ given by the formula

$$\overline{u}(c) = k_1 \left(1 + \tfrac{1}{1-\underline{\rho}_{\widehat{u}}}\, c^{1-\underline{\rho}_{\widehat{u}}}\right).$$

Moreover standard considerations show that there exists $k_2 > 0$ such that the value function $\overline{v} : [0, \infty) \to \mathbb{R}$ of an exponential consumer with utility function $\overline{u}$ is dominated by a function of the form

$$x \mapsto k_2 \left(1 + \tfrac{1}{1-\underline{\rho}_{\widehat{u}}}\, x^{1-\underline{\rho}_{\widehat{u}}}\right).$$

In particular, $\overline{v}$ cannot grow linearly. Now $v \leq \overline{v}$, since $\widehat{u}_0, \widehat{u}_+ \leq \overline{u}$. Hence v cannot grow linearly either. We must therefore have $v'' < 0$ on $(0, \infty)$. ■

A.9. Proof of Theorem 23. We begin by applying Itô's Lemma to m to obtain

$$dm = (\tfrac{1}{2}\sigma^2 x^2 m'' + (\mu x + y - c) m')dt + \sigma x m' dz.$$

Next, we put $\widetilde{c} = f(m)$. Then, differentiating equation (11) with respect to x, we have

$$\tfrac{1}{2}\sigma^2 x^2 m'' + (\mu x + y - \widetilde{c}) m' - \gamma m + \sigma^2 x m' + \mu m - \widetilde{c}' m + \beta u'(\widetilde{c}) \widetilde{c}' = 0$$

when $x > 0$. Moreover this equality extends by continuity to the case $x = 0$. Hence

$$\begin{aligned}
\tfrac{1}{2}\sigma^2 x^2 m'' + (\mu x + y - c) m' &= \tfrac{1}{2}\sigma^2 x^2 m'' + (\mu x + y - \widetilde{c}) m' + (\widetilde{c} - c) m' \\
&= \gamma m - \sigma^2 x m' - \mu m + \widetilde{c}' m - \beta u'(\widetilde{c}) \widetilde{c}' + (\widetilde{c} - c) m' \\
&= \gamma m - \sigma^2 x m' - \mu m + \widetilde{c}' m - \beta m \widetilde{c}' + (\widetilde{c} - c) m' \\
&= (\gamma - \mu + (1 - \beta) \widetilde{c}') m - (\sigma^2 x - (\widetilde{c} - c)) m'
\end{aligned}$$

and

$$\begin{aligned}\frac{dm}{m} &= \left(\gamma - \mu + (1-\beta)\,\widetilde{c}' - \sigma^2\,x\,\frac{m'}{m} + (\widetilde{c} - c)\,\frac{m'}{m}\right)dt + \sigma\,x\,\frac{m'}{m}\,dz \\ &= \left(\gamma - \mu + (1-\beta)\,\widetilde{c}' + \sigma^2\,\rho(\widetilde{c})\,\frac{x\,\widetilde{c}'}{\widetilde{c}} - (\widetilde{c} - c)\,\rho(\widetilde{c})\,\frac{\widetilde{c}'}{\widetilde{c}}\right)dt - \sigma\,\rho(\widetilde{c})\,\frac{x\,\widetilde{c}'}{\widetilde{c}}\,dz\end{aligned}$$

since

$$\frac{m'}{m} = \frac{u''(\widetilde{c})\,\widetilde{c}'}{u'(\widetilde{c})} = \frac{\widetilde{c}\,u''(\widetilde{c})}{u'(\widetilde{c})}\,\frac{\widetilde{c}'}{\widetilde{c}} = -\rho(\widetilde{c})\,\frac{\widetilde{c}'}{\widetilde{c}}.$$

In particular, we have the first statement of the theorem.

As for the second statement, note that if $x = 0$ and $c(0+) > c(0)$ then $c(0) = y$. Wealth therefore remains constant at 0 forever, $m = u'(y)$ forever and $dm = 0$. ■

A.10. Proof of Theorem 27. We proceed in steps. First, put $X = \log(\mu\,x + y)$, $v(x) = (\mu\,x + y)^{1-\rho}\,V(X)$, $\ell = \log(y)$ and, in a slight abuse of notation,

$$h_0(\phi, y) = \left\{\begin{array}{ll} u(y) - \phi\,y & \text{if } \phi \in \big(-\infty, \frac{1}{\beta}\,u'(y)\big] \\ h_+(\phi) & \text{if } \phi \in \big(\frac{1}{\beta}\,u'(y), \infty\big)\end{array}\right\}.$$

Then

$$0 = (\mu\,x + y)\,v' - \gamma\,v + h_+(v')$$

(using the Bellman equation of the deterministic IG model, namely (29))

$$= (\mu\,x + y)^{1-\rho}\,(\mu\,V' + (1-\rho)\,\mu\,V - \gamma\,V) + h_+((\mu\,x + y)^{-\rho}\,(\mu\,V' + (1-\rho)\,\mu\,V))$$

(since $v' = (\mu\,x + y)^{-\rho}\,(\mu\,V' + (1-\rho)\,\mu\,V)$)

$$= (\mu\,x + y)^{1-\rho}\,\left(\mu\,V' + (1-\rho)\,\mu\,V - \gamma\,V + h_+(\mu\,V' + (1-\rho)\,\mu\,V)\right)$$

(because h_+ is homogeneous of degree $1 - \frac{1}{\rho}$). Also

$$0 = y\,v'(0) - \gamma\,v(0) + h_0(v'(0), y)$$

(using the boundary condition of the deterministic IG model, namely (30))

$$= y^{1-\rho}\,(\mu\,V'(\ell) + (1-\rho)\,\mu\,V(\ell) - \gamma\,V(\ell)) + h_0(y^{-\rho}\,(\mu\,V'(\ell) + (1-\rho)\,\mu\,V(\ell)), y)$$

(since $v'(0) = y^{-\rho}\,(\mu\,V'(\ell) + (1-\rho)\,\mu\,V(\ell)))$

$$= y^{1-\rho}\,\left(\mu\,V'(\ell) + (1-\rho)\,\mu\,V(\ell) - \gamma\,V(\ell) + h_0(\mu\,V'(\ell) + (1-\rho)\,\mu\,V(\ell), 1)\right).$$

Hence v is the value function of the deterministic IG model iff V satisfies the Bellman equation

$$0 = \mu\,V' + (1-\rho)\,\mu\,V - \gamma\,V + h_+(\mu\,V' + (1-\rho)\,\mu\,V) \tag{31}$$

for $X \in (\ell, \infty)$ with boundary condition

$$0 = \mu\,V' + (1-\rho)\,\mu\,V - \gamma\,V + h_0(\mu\,V' + (1-\rho)\,\mu\,V, 1) \tag{32}$$

at $X = \ell$.

Second, put $\zeta = \mu\,V' + (1-\rho)\,\mu\,V$. Then equation (31) determines a curve C_+ in (V', V)-space parametrized by $\zeta \in (0, \infty)$. If $\rho < 1$, then: for $\zeta \in (0, \widehat{u}'_+(1))$, V' is increasing in ζ and V is decreasing in ζ; and, for $\zeta \in (\widehat{u}'_+(1), \infty)$, both V' and V are increasing in ζ. If $\rho > 1$, and if we put

$$a = \frac{\gamma - (1-\rho)\,\mu}{(\rho - 1)\,\mu} > 1,$$

then: for $\zeta \in (0, \widehat{u}'_+(a))$, both V' and V are decreasing in ζ;[29] for $\zeta \in (\widehat{u}'_+(a), \widehat{u}'_+(1))$, V' is increasing in ζ and V is decreasing in ζ; and, for $\zeta \in (\widehat{u}'_+(1), \infty)$, both V' and V are increasing in ζ.

Third, V is minimized when $\zeta = \widehat{u}'_+(1)$, at which point $V = \widehat{u}_+(1)$. Hence there are two points on C_+ at which $V = u(1) > \widehat{u}_+(1)$. We denote the corresponding values of ζ by $\widehat{u}'_+(\overline{\psi})$ and $\widehat{u}'_+(\psi)$.[30] It is easy to verify that

$$\psi = \frac{\beta + \rho - 1}{\rho} < 1 < \overline{\psi},$$

but there is no closed-form expression for $\overline{\psi}$.

Fourth, equation (32) determines a curve C_0 in (V', V)-space parametrized by $\zeta \in (-\infty, \infty)$. For $\zeta \in (-\infty, \widehat{u}'_+(\psi))$, V' is increasing in ζ and V is constant and equal to $u(1)$; and, for $\zeta \in (\widehat{u}'_+(\psi), \infty)$, C_0 coincides with C_+ (in particular, both V' and V are increasing in ζ).

[29] Since $V, V' \to 0-$ as $\zeta \to 0+$, we have $V' < 0$ for $\zeta \in (0, \widehat{u}'_+(A))$.

[30] The points ψ and $\overline{\psi}$ are the two solutions of the equation $\widehat{u}_+(\widehat{c}) + (1-\widehat{c})\,\widehat{u}'_+(\widehat{c}) = u(1)$.

Fifth, there is a unique point on C_+ at which $V' = 0$. We denote the corresponding value of ζ by $\widehat{u}'_+(b)$. It is easy to verify that

$$b = \frac{\gamma - (1-\rho)\,\mu}{\rho\,\mu}.$$

It can also be shown that $V' = \frac{\gamma - \beta\,\mu}{\gamma\,\mu}\,\widehat{u}'_+(\psi)$ when $\zeta = \widehat{u}'_+(\psi)$, $V' = \frac{\gamma-\mu}{\gamma\,\mu}\,\widehat{u}'_+(1)$ when $\zeta = \widehat{u}'_+(1)$, $V' = \frac{\gamma-(1-\rho)\,\mu-\rho\,\mu\,\overline{\psi}}{\gamma\,\mu}\,\widehat{u}'_+(\overline{\psi})$ when $\zeta = \widehat{u}'_+(\overline{\psi})$, and, if $\rho > 1$, then $V' = \frac{\gamma-(1-\rho)\,\mu}{\gamma\,(1-\rho)\,\mu}\,\widehat{u}'_+(a) < 0$ when $\zeta = \widehat{u}'_+(a)$.[31]

Sixth, it is easy to verify that b is increasing in γ. Moreover there exists $\gamma_3 \in (\mu, \infty)$ such that: $b \in (0, \psi)$ iff $\gamma \in ((1-\rho)\,\mu, \beta\,\mu)$; $b \in (\psi, 1)$ iff $\gamma \in (\beta\,\mu, \mu)$; $b \in (1, \overline{\psi})$ iff $\gamma \in (\mu, \gamma_3)$; and $b \in (\overline{\psi}, \infty)$ iff $\gamma \in (\gamma_3, \infty)$.[32]

Seventh, we complete the analysis of APC using a phase diagram. Equation (31) is a first-order autonomous ordinary differential equation. A one-dimensional phase diagram (in V-space) would therefore appear to be appropriate. However, there may be upward jumps in V'.[33] It is therefore preferable to work with a two-dimensional phase diagram (in (V', V)-space). The phase curve corresponding to the equilibrium starts on the curve C_0, is confined to the curve C_+, and converges to the steady state $V' = 0$ as $X \to \infty$.

Eighth, we complete the analysis of c by noting that, when $x > 0$: the first-order condition of the equivalent exponential problem gives $\widehat{u}''_+(\widehat{c})\,\widehat{c}' = v''$; and the Bellman equation of the deterministic IG model gives $0 = (\mu\,x + y - \widehat{c})\,v'' - (\gamma - \mu)\,v'$. We therefore have

$$\widehat{c}' = \frac{(\gamma - \mu)\,v'}{(\mu\,x + y)\,(1 - \widehat{APC})\,\widehat{u}''_+(\widehat{c})},$$

where $\widehat{APC} = \frac{\widehat{c}}{(\mu\,x+y)}$. ■

[31] These values of V' are found from the formula $V' = \frac{\gamma-(1-\rho)\,\mu-\rho\,\mu\,\widehat{c}}{\gamma\,\mu}\,\widehat{u}'_+(\widehat{c})$. The corresponding values of V can be found from the formula $V = \frac{1-\rho+\rho\,\widehat{c}}{\widehat{c}}\,\widehat{u}_+(\widehat{c})$.

[32] The critical value of γ can be found by solving the equation $\frac{\gamma}{\alpha\,\mu} = \left(\frac{\gamma-(1-\rho)\,\mu}{\psi\,\rho\,\mu}\right)^{\rho}$. This equation has two solutions, $\alpha\,\mu$ and γ_3.

[33] There are no downward jumps. Intuitively speaking, this is because V is the upper envelope of smooth functions, and can therefore have convex kinks but not concave kinks.